L^p estimates for the iterated Hardy-Littlewood maximal operator

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Let f be a locally integrable function in the n-dimensional Euclidean space R^n and define the Hardy-Littlewood maximal function as follows: $Mf(x) = \sup|Q|^{-1}\int_Q|f|$, where the sup is taken over all the balls Q centered at x. The kth iteration of M is defined by induction: $M^k f(x) = M(M^{k-1}f)(x)$. The main result of the paper can be formulated as follows: Theorem 2.3: (1) If $1 < p < \infty$, then the L^p norms of f and $M^k f$ are equivalent for $k \geq 1$. Let $f \in L^1(R^n), \operatorname{supp} f \subset B$, where B is a finite ball; then (2) for $p = 1, \int_B M^k f < \infty$ iff $\int_B |f|(\log|f|)^k < \infty$, and (3) for $0 < p < 1$, $\int_B (M^k f)^p < \infty$ iff $\int_B |f|(\log|f|)^{k-1} < \infty$.

The second part of the paper is devoted to the generalizations of these and similar results to the case of linear spaces over local fields.

Abstract

In light of two measure estimate inequalities from [2] for the iterated Hardy-Littlewood maximal operator $M^k f$, certain equivalence between $M^k f$ and the Zygmund class $L\log^a L$ are established on $\mathbb{R}^n$, so that we generalize Stein's $L\log L$ theorem. In Section 3, a simple induction enables us to prove such extensions on K^n, the n-dimensional linear space over a local field K, without recoursing to Leckband's result.

1 Introduction

Using the argument of nonincreasing rearrangement of f, Leckband obtained (for technique reasons, we record as follows):

Theorem L Let $\lambda > 0$ and $k \in \mathbb{N}$. Then there exist constants $A > 1$ and $0 < B < 1$ dependent only on n, k so that

$$(i) \qquad |\{x \in \mathbb{R}^n : M^k f(x) > \lambda\}| \le A \int_{\{|f|>\lambda/A\}} \frac{|f(x)|}{\lambda}(1 + \log^+ \frac{|f(x)|}{\lambda})^{k-1} dx,$$

$$(ii) \qquad |\{x \in \mathbb{R}^n : M^k f(x) > B\lambda\}| \le A \int_{\{|f|>\lambda\}} \frac{|f(x)|}{\lambda}(1+\log^+ \frac{|f(x)|}{B\lambda})^{k-1} dx,$$

where the set $\{g > \lambda\} = \{x \in \mathbb{R}^n : g(x) > \lambda\}$ and the function $\log^+ x = \max\{\log x, 0\}, x \ge 0$.

The following pointwise estimate in Pérez[5] may have intimate relation with Leckband inequality above.

For $k = 1, 2, \ldots,$

$$M^{k+1} f(x) \sim M_{\varphi_k} f(x),$$

where $\varphi_k(t) = t(1 + \log^+ t)^k$.

Let (X, d, μ) be a space of homogeneous type, where d is a quasi-metric on the set X and μ is a doubling Borel measure; moreover, $\mu(B(x, r)) < \infty$, $B(x, r) = \{y \in X : d(x, y) < r\}$.

Macias and Segovia showed that there exists a quasi-metric ρ, generating the same topology as d. Among other properties, there is a $\delta_0 > 0$ such that

$\rho(x,y)^{\delta_0}$ is a metric on X. (When $X = \mathbb{R}^n$, $d(x,y) = |x-y|$, we may take $\rho(x,y) = |x-y|^n$, then $\delta_0 = 1/n$.

Note.

References

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