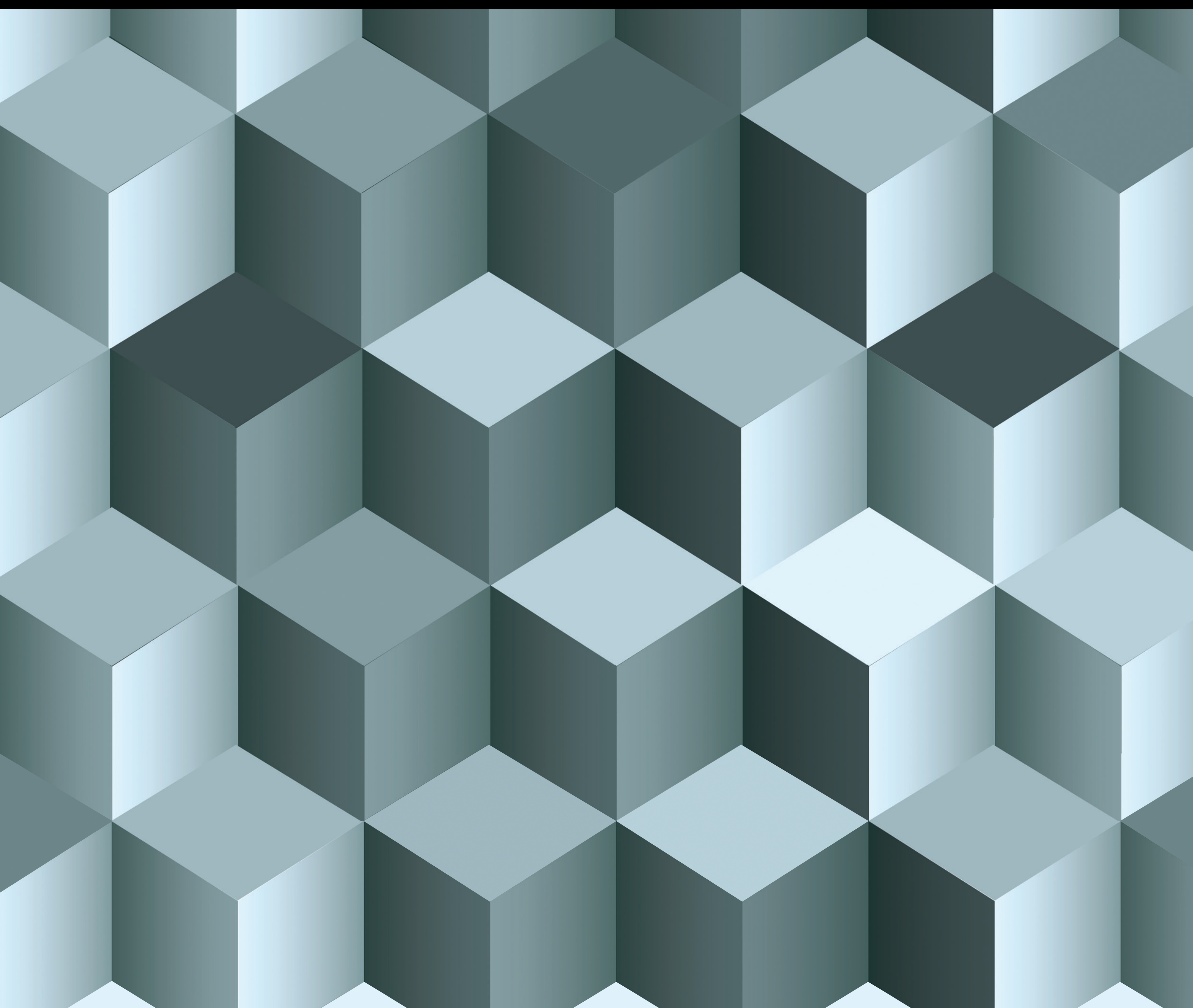
Journal of Function Spaces

Partial Differential Equations and Function Spaces

Guest Editors: Shijun Zheng, Simone Secchi, Huoxiong Wu, and Nguyen Cong Phuc



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Editorial

Partial Differential Equations and Function Spaces

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It is well known that PDEs and the theory of function spaces have played a central role in the mathematical analysis of problems arising from mathematical physics, biology, and other branches of modern applied sciences. This special issue addresses the current advances in these two broad areas; in particular it focuses on the connections and interactions between them. We are happy that this issue has received the attention of active researchers with interesting and valuable contributions in the field. Topics cover areas from existence and nonexistence theorems for degenerate differential operators to multilinear fractional singular integral operators on weighted function spaces, which might show applications of theory of function spaces in understanding the qualitative as well as quantitative behaviors of PDEs.

This special issue consists of five research papers. L. Dai et al. obtain the existence of solutions to an elliptic equation in a bounded domain for a Leray-Lions operator acting from a weighted Sobolev space into its dual. The paper by Q. Wu et al. establishes the nonexistence of solutions to a semilinear elliptic equation with unbounded inhomogeneous power nonlinearity. They show that there exists no solution with finite Morse index in the energy-subcritical regime. This is a Liouville type theorem that has proven to be a powerful tool in obtaining a priori bound for solutions of elliptic equations. The paper by J. Cunanan and Y. Tsutsui deals with trace operators of Wiener amalgam spaces using frequency uniform decomposition operators and maximal inequalities. Wiener amalgam spaces, together with modulation spaces, were introduced in the 1980s and are now widely used for various problems in PDE and harmonic analysis. They resemble Triebel-Lizorkin spaces in the sense that one takes $L^p(\ell^q)$ norms using frequency uniform decomposition operators through unit cubes instead of dyadic annuli. S. He and X. Tao study the boundedness of multilinear fractional integral operators as well as the boundedness of multilinear maximal singular integral operators with rough kernels on weighted Morrey spaces. Y. Xie et al. obtain the existence and the structure of a compact uniform attractor for a nonautonomous diffusion equation with fading memory, by verifying the uniform asymptotic compactness of a family of processes using asymptotic contractive method. These results are proven under the conditions that the nonlinearity satisfies the polynomial growth of arbitrary order and the time dependent forcing term is only translation-bounded.

We hope that the readers who are interested in analysis and applications of function spaces, differential operators, and PDEs will find useful information and perspectives in this special issue. Meanwhile, we also hope that in the near future we can see new papers be published based on this issue.

Acknowledgments

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Shijun Zheng
Simone Secchi
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Nguyen Cong Phuc

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Research Article

On the Theory of Multilinear Singular Operators with Rough Kernels on the Weighted Morrey Spaces

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We study some multilinear operators with rough kernels. For the multilinear fractional integral operators $T^A_{\Omega,\alpha}$ and the multilinear fractional maximal integral operators $M^A_{\Omega,\alpha}$, we obtain their boundedness on weighted Morrey spaces with two weights $L^{p,\kappa}(u,v)$ when $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m-1$) or $D^\gamma A \in \mathrm{BMO}$ ($|\gamma| = m-1$). For the multilinear singular integral operators T^A_Ω and the multilinear maximal singular integral operators M^A_Ω, we show they are bounded on weighted Morrey spaces with two weights $L^{p,\kappa}(u,v)$ if $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m-1$) and bounded on weighted Morrey spaces with one weight $L^{p,\kappa}(w)$ if $D^\gamma A \in \mathrm{BMO}$ ($|\gamma| = m-1$) for $m = 1, 2$.

1. Introduction and Main Results

Let us consider the following multilinear fractional integral operator,

$$T^A_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha+m-1}} R_m(A;x,y) f(y)\,dy, \qquad 0<\alpha<n, \tag{1}$$

and the multilinear fractional maximal operator:

$$M^A_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha+m-1}} \cdot \int_{|x-y|<r} |\Omega(x-y) R_m(A;x,y) f(y)|\,dy, \qquad 0<\alpha<n, \tag{2}$$

where $\Omega \in L^s(S^{n-1})$ $(s>1)$ is homogeneous of degree zero in $\mathbb{R}^n$, A is a function defined on $\mathbb{R}^n$, and $R_m(A;x,y)$ denotes the mth order Taylor series remainder of A at x expanded about y; that is,

$$R_m(A;x,y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma, \tag{3}$$

$\gamma = (\gamma_1,\dots,\gamma_n)$, each γ_i, $i = 1,\dots,n$, is a nonnegative integer, $|\gamma| = \sum_{i=1}^n \gamma_i$, $\gamma! = \gamma_1!\cdots\gamma_n!$, $x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}$, and $D^\gamma = \partial^{|\gamma|}/\partial^{\gamma_1}x_1\cdots\partial^{\gamma_n}x_n$.

We notice that if $\alpha = 0$, the above two operators $T^A_{\Omega,\alpha}$, $M^A_{\Omega,\alpha}$ are the multilinear singular integral operator T^A_Ω and the multilinear maximal singular integral operator M^A_Ω whose definitions are given as follows, respectively:

$$T^A_\Omega f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)\,dy, \tag{4}$$

$$M^A_\Omega f(x) = \sup_{r>0} \frac{1}{r^{n+m-1}} \cdot \int_{|x-y|<r} |\Omega(x-y) R_m(A;x,y) f(y)|\,dy. \tag{5}$$

For $m=1$, $T^A_{\Omega,\alpha}$ is obviously the commutator $[A, T_{\Omega,\alpha}]$ of $T_{\Omega,\alpha}$ and A: $[A, T_{\Omega,\alpha}]f(x) = A(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(Af)(x)$, where $T_{\Omega,\alpha}$ is the fractional integral operator given by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\,dy, \quad 0<\alpha<n. \tag{6}$$

There are numerous works on the study of multilinear operators with rough kernels. If $D^\gamma A \in \mathrm{BMO}$ ($|\gamma| = m-1$), the L^p boundedness of T_Ω^A was obtained by means of a good-λ inequality by Cohen and Gosselin [1]. In 1994, Hofmann [2] proved that T_Ω^A is a bounded operator on $L^p(w)$ when $\nabla A \in BMO$ and $w \in A_p$. Recently, Lu et al. [3] proved T_Ω^A and M_Ω^A are bounded from L^p to L^q $(1/p - 1/q = \beta/n)$ when $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m-1)$, while for multilinear fractional integral operators, Ding and Lu [4] showed the $(L^p(w^p), L^q(w^q))$ boundedness of $T_{\Omega,\alpha}^{A_1,\ldots,A_k}$ and $M_{\Omega,\alpha}^{A_1,\ldots,A_k}$ (their definitions will be given later) if $D^\gamma A_j \in \mathrm{BMO}$ $(|\gamma| = m-1)$, $j = 1, \ldots, k$. After that, Lu and Zhang [5] proved $T_{\Omega,\alpha}^A$ is a bounded operator from L^p to L^q $(1/p - 1/q = (\alpha+\beta)/n)$ when $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m-1)$.

On the other hand, the classical Morrey spaces were first introduced by Morrey [6] to study the local behavior of solutions to second-order elliptic partial differential equations. From then on, a lot of works concerning Morrey spaces and some related spaces have been done; see [7–9] and the references therein for details. In 2009, Komori and Shirai [10] first studied the weighted Morrey spaces and investigated some classical singular integrals in harmonic analysis on them, such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator, the fractional integral operator, and the fractional maximal operator. Recently, Wang [11] discussed the boundedness of the classical singular operators with rough kernels on the weighted Morrey spaces.

We note that many works concerning $T_{\Omega,\alpha}^A$, $M_{\Omega,\alpha}^A$, T_Ω^A, and M_Ω^A have been done on L^p spaces or weighted L^p spaces when $D^\gamma A$ belongs to some function spaces for $|\gamma| = m-1$. However, there is not any study about these operators on weighted Morrey spaces. Therefore, it is natural to ask whether they are bounded on weighted Morrey spaces. The aim of this paper is to investigate the boundedness of $T_{\Omega,\alpha}^A$, $M_{\Omega,\alpha}^A$, T_Ω^A, and M_Ω^A on weighted Morrey spaces if $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m-1)$ or $D^\gamma A \in \mathrm{BMO}$ $(|\gamma| = m-1)$. When $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m-1)$, we show $T_{\Omega,\alpha}^A$ and T_Ω^A are controlled pointwisely by the fractional singular integral operators $\overline{T}_{\Omega,\alpha+\beta}$ and $\overline{T}_{\Omega,\beta}$ (their definition will be given later), respectively. Thus, the problem of studying the boundedness of $T_{\Omega,\alpha}^A$ and T_Ω^A on weighted Morrey spaces with two weights could be reduced to that of $\overline{T}_{\Omega,\alpha+\beta}$ and $\overline{T}_{\Omega,\beta}$. When $D^\gamma A \in \mathrm{BMO}$ $(|\gamma| = m-1)$, the boundedness of $T_{\Omega,\alpha}^A$ on weighted Morrey spaces with two weights is proved by standard method. However, we could only obtain the boundedness of T_Ω^A on weighted Morrey spaces with one weight for $m = 1$ and $m = 2$, since we need the $L^p(w)$ boundedness of T_Ω^A in our proof, but to the best of our knowledge, there is not such bounds hold for T_Ω^A when $m \geq 3$. For $M_{\Omega,\alpha}^A$ and M_Ω^A, we show they are controlled pointwisely by $T_{\Omega,\alpha}^A$ and T_Ω^A, respectively. Thus, it is easy to obtain the same results for $M_{\Omega,\alpha}^A$ and M_Ω^A as those of $T_{\Omega,\alpha}^A$ and T_Ω^A.

Before stating our main results, we introduce some definitions and notations at first.

A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$, the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E. The weighted Lebesgue spaces with respect to the measure $w(x)dx$ are denoted by $L^p(w)$ with $0 < p < \infty$. We say a weight w satisfies the doubling condition if there exists a constant $D > 0$ such that for any ball B, we have $w(2B) \leq D w(B)$. When w satisfies this condition, we denote $w \in \Delta_2$ for short.

Throughout this paper, $B(x_0, r)$ denotes a ball centered at x_0 with radius r. Let Q be a cube with sides parallel to the axes. For $K > 0$, KQ denotes the cube with the same center as Q and side-length being K times longer. When $\alpha = 0$, we will denote $T_{\Omega,\alpha}$, $T_{\Omega,\alpha}^A$, $M_{\Omega,\alpha}^A$ by T_Ω, T_Ω^A, M_Ω^A, respectively. And for any number a, a' stands for the conjugate of a. The letter C denotes a positive constant that may vary at each occurrence but is independent of the essential variable.

Next, we give the definition of weighted Morrey space introduced in [10].

Definition 1. Let $1 \leq p < \infty$, let $0 < \kappa < 1$, and let w be a weight. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(w) := \left\{ f \in L^p_{\mathrm{loc}}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty \right\}, \tag{7}$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x)\, dx \right)^{1/p}, \tag{8}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

When we investigate the boundedness of the multilinear fractional integral operator, we need to consider the weighted Morrey space with two weights. It is defined as follows.

Definition 2. Let $1 \leq p < \infty$, let $0 < \kappa < 1$, and let u, v be two weights. The two weights weighted Morrey space is defined by

$$L^{p,\kappa}(u,v) := \left\{ f : \|f\|_{L^{p,\kappa}(u,v)} < \infty \right\}, \tag{9}$$

where

$$\|f\|_{L^{p,\kappa}(u,v)} = \sup_B \left(\frac{1}{v(B)^\kappa} \int_B |f(x)|^p u(x)\, dx \right)^{1/p}, \tag{10}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$. If $u = v$, then we denote $L^{p,\kappa}(u)$ for short.

As is pointed out in [10], we could also define the weighted Morrey spaces with cubes instead of balls. So we shall use these two definitions of weighted Morrey spaces appropriate to calculation.

Then, we give the definitions of Lipschitz space and BMO space.

Definition 3. The Lipschitz space of order β, $0 < \beta < 1$, is defined by

$$\dot{\Lambda}_\beta(\mathbb{R}^n) = \left\{ f : |f(x) - f(y)| \leq C |x-y|^\beta \right\}, \tag{11}$$

and the smallest constant $C > 0$ is the Lipschitz norm $\|\cdot\|_{\dot{\Lambda}_\beta}$.

Definition 4. A locally integrable function b is said to be in $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|b\|_* = \|b\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B|\, dx < \infty, \quad (12)$$

where

$$b_B = \frac{1}{|B|} \int_B b(y)\, dy, \quad (13)$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

At last, we give the definition of two weight classes.

Definition 5. A weight function w is in the Muckenhoupt class A_p with $1 < p < \infty$ if there exists $C > 1$ such that for any ball B,

$$\left(\frac{1}{|B|} \int_B w(x)\, dx\right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)}\, dx\right)^{p-1} \le C. \quad (14)$$

We define $A_\infty = \bigcup_{1<p<\infty} A_p$.

When $p = 1$, we define $w \in A_1$ if there exists $C > 1$ such that for almost every x,

$$Mw(x) \le Cw(x). \quad (15)$$

Definition 6. A weight function w belongs to $A(p,q)$ for $1 < p < q < \infty$ if there exists $C > 1$ such that such that for any ball B,

$$\left(\frac{1}{|B|} \int_B w(x)^q\, dx\right)^{1/q} \cdot \left(\frac{1}{|B|} \int_B w(x)^{-p/(p-1)}\, dx\right)^{(p-1)/p} \le C. \quad (16)$$

When $p = 1$, then we define $w \in A(1,q)$ with $1 < q < \infty$ if there exists $C > 1$ such that

$$\left(\frac{1}{|B|} \int_B w(x)^q\, dx\right)^{1/q} \left(\operatorname*{ess\,sup}_{x \in B} \frac{1}{w(x)}\right) \le C. \quad (17)$$

Remark 7 (see [10]). If $w \in A(p,q)$ with $1 < p < q$, then

(a) $w^q, w^{-p'}, w^{-q'} \in \Delta_2$.

(b) $w^{-p'} \in A_{t'}$ with $t = 1 + q/p'$.

Now we state the main results of this paper.

Theorem 8. *If $0 < \alpha + \beta < n$, $\Omega \in L^s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero, $1 < s' < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$, $0 < \kappa < p/q$, $w^{s'} \in A(p/s', q/s')$, $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m - 1)$, then*

$$\left\|T^A_{\Omega,\alpha} f\right\|_{L^{q,\kappa q/p}(w^q)} \le C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p,\kappa}(w^p,w^q)}, \quad (18)$$

$$\left\|M^A_{\Omega,\alpha} f\right\|_{L^{q,\kappa q/p}(w^q)} \le C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p,\kappa}(w^p,w^q)}. \quad (19)$$

Theorem 9. *If $0 < \beta < 1$, $\Omega \in L^s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero, $1 < s' < p < n/\beta$, $1/q = 1/p - \beta/n$, $0 < \kappa < p/q$, $w^{s'} \in A(p/s', q/s')$, $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m - 1)$, then*

$$\left\|T^A_{\Omega} f\right\|_{L^{q,\kappa q/p}(w^q)} \le C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p,\kappa}(w^p,w^q)}, \quad (20)$$

$$\left\|M^A_{\Omega} f\right\|_{L^{q,\kappa q/p}(w^q)} \le C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p,\kappa}(w^p,w^q)}. \quad (21)$$

Theorem 10. *If $0 < \alpha < n$, $\Omega \in L^s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero, $1 < s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$, $w^{s'} \in A(p/s', q/s')$, $D^\gamma A \in \mathrm{BMO}$ $(|\gamma| = m - 1)$, then*

$$\left\|T^A_{\Omega,\alpha} f\right\|_{L^{q,\kappa q/p}(w^q)} \le C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_* \|f\|_{L^{p,\kappa}(w^p,w^q)}, \quad (22)$$

$$\left\|M^A_{\Omega,\alpha} f\right\|_{L^{q,\kappa q/p}(w^q)} \le C \sum_{|\gamma|=m-1} \left\|D^\gamma A\right\|_* \|f\|_{L^{p,\kappa}(w^p,w^q)}. \quad (23)$$

When $m = 1$ and $m = 2$, we denote T^A_Ω, M^A_Ω by $[A, T_\Omega]$, $[A, M_\Omega]$, and $\widetilde{T}^A_\Omega$, $\widetilde{M}^A_\Omega$, respectively, in order to distinguish from T^A_Ω and M^A_Ω that are defined for any $m \in \mathbb{N}^*$. To be more precise,

$$\begin{aligned}
[A, T_\Omega] f(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (A(x) - A(y)) \cdot f(y)\, dy, \\
[A, M_\Omega] f(x) &= \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} \Omega(x-y) \cdot (A(x) - A(y)) f(y)\, dy, \\
\widetilde{T}^A_\Omega f(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} (A(x) - A(y) - \nabla A(y) \cdot (x-y)) f(y)\, dy, \\
\widetilde{M}^A_\Omega f(x) &= \sup_{r>0} \frac{1}{r^{n+1}} \int_{|x-y|<r} \Omega(x-y) \cdot (A(x) - A(y) - \nabla A(y)(x-y)) f(y)\, dy.
\end{aligned} \quad (24)$$

Then for the above operators, we have the following results on weighted Morrey spaces with one weight.

Theorem 11. *If $\Omega \in L^s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero and satisfies the vanishing condition $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$, $1 < s' < p < \infty$, $0 < \kappa < 1$, $w \in A_{p/s'}$, $A \in \mathrm{BMO}$, then*

$$\left\|[A, T_\Omega] f\right\|_{L^{p,\kappa}(w)} \le C \|A\|_* \|f\|_{L^{p,\kappa}(w)}, \quad (25)$$

$$\left\|[A, M_\Omega] f\right\|_{L^{p,\kappa}(w)} \le C \|A\|_* \|f\|_{L^{p,\kappa}(w)}. \quad (26)$$

Theorem 12. *If $\Omega \in L^{\infty}(S^{n-1})$ is homogeneous of degree zero and satisfies the moment condition $\int_{S^{n-1}} \theta\Omega(\theta)d\theta = 0$, $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $\nabla A \in \mathrm{BMO}$, then*

$$\left\| \widetilde{T}_{\Omega}^{A} f \right\|_{L^{p,\kappa}(w)} \leq C \|\nabla A\|_{*} \|f\|_{L^{p,\kappa}(w)}, \quad (27)$$

$$\left\| \widetilde{M}_{\Omega}^{A} f \right\|_{L^{p,\kappa}(w)} \leq C \|\nabla A\|_{*} \|f\|_{L^{p,\kappa}(w)}. \quad (28)$$

Remark 13. Here we point out that for T_{Ω}^{A} and M_{Ω}^{A}, when $D^{\gamma}A \in \mathrm{BMO}$ ($|\gamma| = m-1$), the analogues of Theorems 11 and 12 are open for $m \geq 3$.

Remark 14. Define

$$\begin{aligned} T_{\Omega,\alpha}^{A_1,\ldots,A_k} f(x) &= \int_{\mathbb{R}^n} \prod_{i=1}^{k} R_{m_i}(A_i; x, y) \\ &\quad \cdot \frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}} f(y)\, dy, \\ M_{\Omega,\alpha}^{A_1,\ldots,A_k} f(x) &= \sup_{r>0} \frac{1}{r^{n-\alpha+N}} \int_{|x-y|<r} |\Omega(x-y)| \\ &\quad \cdot \prod_{i=1}^{k} \left| R_{m_i}(A_i; x, y) \right| |f(y)|\, dy, \end{aligned} \quad (29)$$

where $R_{m_i}(A_i; x, y) = A_i(x) - \sum_{|\gamma|<m_i} (1/\gamma!) D^{\gamma} A_i(y)(x-y)^{\gamma}$, $i = 1, \ldots, k$, $N = \sum_{i=1}^{k}(m_i - 1)$. When $0 < \alpha < n$, they are a class of multilinear fractional integral operators and multilinear fractional maximal operators. When $\alpha = 0$, they are a class of multilinear singular integral operators and multilinear maximal singular integral operators. Repeating the proofs of the theorems above, we will find that for $T_{\Omega,\alpha}^{A_1,\ldots,A_k}$ and $M_{\Omega,\alpha}^{A_1,\ldots,A_k}$, the conclusions of Theorems 8 and 9 above with the bounds $C\prod_{i=1}^{k}(\sum_{|\gamma|=m_i-1} \|D^{\gamma}A_i\|_{\dot{\Lambda}_{\beta}})$ and Theorem 10 with the bounds $C\prod_{i=1}^{k}(\sum_{|\gamma|=m_i-1} \|D^{\gamma}A_i\|_{*})$ also hold, respectively.

The organization of this paper is as follows. In Section 2, we give some requisite lemmas and well-known results that are important in proving the theorems. The proof of the theorems will be shown in Section 3.

2. Lemmas and Well-Known Results

Lemma 15 (see [1]). *Let $A(x)$ be a function on $\mathbb{R}^n$ with mth order derivatives in $L^{l}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $l > n$. Then*

$$\begin{aligned} &\left| R_m(A; x, y) \right| \\ &\quad \leq C |x-y|^{m} \sum_{|\gamma|=m} \left(\frac{1}{|I_x^{\gamma}|} \int_{I_x^{\gamma}} \left| D^{\gamma} A(z) \right|^{l} dz \right)^{1/l}, \end{aligned} \quad (30)$$

where I_x^{γ} is the cube centered at x with sides parallel to the axes, whose diameter is $5\sqrt{n}|x-y|$.

Lemma 16 (see [12]). *For $0 < \beta < 1$, $1 \leq q < \infty$, we have*

$$\begin{aligned} \|f\|_{\dot{\Lambda}_{\beta}} &\approx \sup_{Q} \frac{1}{|Q|^{1+\beta/n}} \int_{Q} \left| f(x) - b_Q(f) \right| dx \\ &\approx \sup_{Q} \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_{Q} \left| f(x) - b_Q(f) \right|^{q} dx \right)^{1/q}. \end{aligned} \quad (31)$$

For $q = \infty$, the formula should be interpreted appropriately.

Lemma 17 (see [13]). *Let $Q_1 \subset Q_2$, $g \in \dot{\Lambda}_{\beta}$ $(0 < \beta < 1)$. Then*

$$\left| g_{Q_1} - g_{Q_2} \right| \leq C |Q_2|^{\beta/n} \|g\|_{\dot{\Lambda}_{\beta}}. \quad (32)$$

Theorem 18 (see [14]). *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $\Omega \in L^{s}(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero. Then $T_{\Omega,\alpha}$ is a bounded operator from $L^{p}(w^{p})$ to $L^{q}(w^{q})$, if the index set $\{\alpha, p, q, s\}$ satisfies one of the following conditions:*

(a) *$s' < p$ and $w(x)^{s'} \in A(p/s', q/s')$;*

(b) *$s > q$ and $w(x)^{-s'} \in A(q'/s', p'/s')$;*

(c) *$\alpha/n + 1/s < 1/p < 1/s'$ and there is r, $1 < r < s/(n/\alpha)'$ such that $w(x)^{r'} \in A(p, q)$.*

Lemma 19 (see [10]). *If $w \in \Delta_2$, then there exists a constant $D_1 > 1$, such that*

$$w(2B) \geq D_1 w(B). \quad (33)$$

We call D_1 the reverse doubling constant.

Theorem 20 (see [4]). *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $\Omega \in L^{s}(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero. Moreover, for $1 \leq i \leq k$, $|\gamma| = m_i - 1$, $m_i \geq 2$, and $D^{\gamma}A_i \in \mathrm{BMO}(\mathbb{R}^n)$, if the index set $\{\alpha, p, q, s\}$ satisfies one of the following conditions:*

(a) *$s' < p$ and $w(x)^{s'} \in A(p/s', q/s')$;*

(b) *$s > q$ and $w(x)^{-s'} \in A(q'/s', p'/s')$;*

(c) *$\alpha/n + 1/s < 1/p < 1/s'$ and there is r, $1 < r < s/(n/\alpha)'$, such that $w(x)^{r'} \in A(p, q)$.*

Then there is a $C > 0$, independent of f and A_i, such that

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \left| T_{\Omega,\alpha}^{A_1,\ldots,A_k} f(x)\, w(x) \right|^{q} dx \right)^{1/q} \\ &\quad \leq C \prod_{i=1}^{k} \left(\sum_{|\gamma|=m_i-1} \left\| D^{\gamma} A_i \right\|_{*} \right) \\ &\quad \cdot \left(\int_{\mathbb{R}^n} |f(x)\, w(x)|^{p} dx \right)^{1/p}. \end{aligned} \quad (34)$$

Lemma 21 (see [15]). *(a) (John-Nirenberg Lemma) Let $1 \leq p < \infty$. Then $b \in \mathrm{BMO}$ if and only if*

$$\frac{1}{|Q|} \int_{Q} |b - b_Q|^{p} dx \leq C \|b\|_{*}^{p}. \quad (35)$$

(b) Assume $b \in$ BMO; then for cubes $Q_1 \subset Q_2$,

$$\left|b_{Q_1} - b_{Q_2}\right| \leq C \log\left(\frac{|Q_2|}{|Q_1|}\right) \|b\|_* . \tag{36}$$

(c) If $b \in$ BMO, then

$$\left|b_{2^{j+1}B} - b_B\right| \leq 2^n (j+1) \|b\|_* . \tag{37}$$

Theorem 22 (see [16]). *Suppose that $\Omega \in L^s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero and satisfies the vanishing condition $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, T_\Omega]$ is bounded on $L^p(w)$ if the index set $\{p, q, s\}$ satisfies one of the following conditions:*

(a) $s' \leq p < \infty$, $p \neq 1$ *and* $w \in A_{p/s'}$;

(b) $1 \leq p \leq s$, $p \neq \infty$ *and* $w^{1-p'} \in A_{p'/s'}$;

(c) $1 \leq p < \infty$ *and* $w^{s'} \in A_p$.

Theorem 23 (see [2]). *If $\Omega \in L^\infty(S^{n-1})$ is homogeneous of degree zero and satisfies the moment condition $\int_{S^{n-1}} \theta\Omega(\theta)d\theta = 0$, $w \in A_p$, $1 < p < \infty$, $\nabla A \in$ BMO, then we have*

$$\left\|\widetilde{T}_\Omega^A f\right\|_{L^p(w)} \leq C \|\Omega\|_\infty \|\nabla A\|_* \|f\|_{L^p(w)} . \tag{38}$$

Lemma 24 (see [15]). *The following are true:*

(1) *If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. More precisely, for all $\lambda > 1$ we have*

$$w(\lambda Q) \leq C\lambda^{np} w(Q) . \tag{39}$$

(2) *If $w \in A_p$ for some $1 \leq p < \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any cube Q and a measurable set $S \subset Q$,*

$$\frac{w(S)}{w(Q)} \leq C\left(\frac{|S|}{|Q|}\right)^\delta . \tag{40}$$

Lemma 25 (see [17]). *Let $w \in A_\infty$. Then the norm of $\mathrm{BMO}(w)$ is equivalent to the norm of $\mathrm{BMO}(\mathbb{R}^n)$, where*

$$\mathrm{BMO}(w) = \left\{b : \|b\|_{*,w} = \sup_Q \frac{1}{w(Q)} \int_Q \left|b(x) - m_{Q,w}b\right| w(x)\,dx\right\}, \tag{41}$$

$$m_{Q,w}b = \frac{1}{w(Q)} \int_Q b(x) w(x)\,dx.$$

3. Proofs of the Main Results

Before proving Theorem 8, we give a pointwise estimate of $T^A_{\Omega,\alpha} f(x)$ at first. Set

$$\overline{T}_{\Omega,\alpha+\beta} f(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)|\,dy, \quad 0 < \alpha + \beta < n, \tag{42}$$

where $\Omega \in L^s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero in $\mathbb{R}^n$. Then we have the following estimate.

Theorem 26. *If $\alpha \geq 0$, $0 < \alpha+\beta < n$, $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m-1)$, then there exists a constant C independent of f such that*

$$\left|T^A_{\Omega,\alpha} f(x)\right| \leq C\left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \overline{T}_{\Omega,\alpha+\beta} f(x) . \tag{43}$$

Proof. For fixed $x \in \mathbb{R}^n$, $r > 0$, let Q be a cube with center at x and diameter r. Denote $Q_k = 2^k Q$ and set

$$A_{Q_k}(y) = A(y) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} m_{Q_k}(D^\gamma A)\, y^\gamma, \tag{44}$$

where $m_{Q_k} f$ is the average of f on Q_k. Then we have, when $|\gamma| = m-1$,

$$D^\gamma A_{Q_k}(y) = D^\gamma A(y) - m_{Q_k}(D^\gamma A), \tag{45}$$

and it is proved in [1] that

$$R_m(A; x, y) = R_m\left(A_{Q_k}; x, y\right). \tag{46}$$

Hence,

$$\begin{aligned} &\left|T^A_{\Omega,\alpha} f(x)\right| \\ &\leq \sum_{k=-\infty}^{\infty} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{\left|R_m\left(A_{Q_k}; x, y\right)\right|}{|x-y|^{m-1}} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\,dy \\ &:= \sum_{k=-\infty}^{\infty} T_k. \end{aligned} \tag{47}$$

By Lemma 15 we get

$$\begin{aligned} &\left|R_m\left(A_{Q_k}; x, y\right)\right| \\ &\leq \left|R_{m-1}\left(A_{Q_k}; x, y\right)\right| \\ &\quad + C \sum_{|\gamma|=m-1} \left|D^\gamma A_{Q_k}(y)\right| |x-y|^{m-1} \\ &\leq C|x-y|^{m-1} \sum_{|\gamma|=m-1} \left(\frac{1}{|I^y_x|} \int_{I^y_x} \left|D^\gamma A_{Q_k}(z)\right|^l dz\right)^{1/l} \\ &\quad + C|x-y|^{m-1} \sum_{|\gamma|=m-1} \left|D^\gamma A_{Q_k}(y)\right|. \end{aligned} \tag{48}$$

Note that, if $|x-y|<2^k r$, then $I_x^y \subset 5nQ_k$. By Lemmas 16 and 17 we have, when $|\gamma|=m-1$,

$$\begin{aligned}
&\left(\frac{1}{|I_x^y|}\int_{I_x^y}\left|D^\gamma A_{Q_k}(z)\right|^l dz\right)^{1/l} \\
&= \left(\frac{1}{|I_x^y|}\int_{I_x^y}\left|D^\gamma A(z)-m_{Q_k}(D^\gamma A)\right|^l dz\right)^{1/l} \\
&\leq \left(\frac{1}{|I_x^y|}\int_{I_x^y}\left|D^\gamma A(z)-m_{I_x^y}(D^\gamma A)\right|^l dz\right)^{1/l} \\
&\quad + \left|m_{I_x^y}(D^\gamma A)-m_{5nQ_k}(D^\gamma A)\right| \\
&\quad + \left|m_{5nQ_k}(D^\gamma A)-m_{Q_k}(D^\gamma A)\right| \\
&\leq C|Q_k|^{\beta/n}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \leq C\left(2^k r\right)^\beta \|D^\gamma A\|_{\dot{\Lambda}_\beta}.
\end{aligned} \tag{49}$$

It is obvious that when $|\gamma|=m-1$,

$$\begin{aligned}
\left|D^\gamma A_{Q_k}(y)\right| &= \left|D^\gamma A(y)-m_{Q_k}(D^\gamma A)\right| \\
&\leq C|Q_k|^{\beta/n}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
&\leq C\left(2^k r\right)^\beta \|D^\gamma A\|_{\dot{\Lambda}_\beta}.
\end{aligned} \tag{50}$$

Thus,

$$\begin{aligned}
&\left|R_m\left(A_{Q_k};x,y\right)\right| \\
&\quad \leq C|x-y|^{m-1}\left(2^k r\right)^\beta \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}.
\end{aligned} \tag{51}$$

Therefore,

$$\begin{aligned}
T_k &\leq C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
&\quad \cdot \int_{2^{k-1}r\leq|x-y|<2^k r}\frac{\left(2^k r\right)^\beta}{|x-y|^{n-\alpha}}|\Omega(x-y)||f(y)|\,dy \\
&\leq C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
&\quad \cdot \int_{2^{k-1}r\leq|x-y|<2^k r}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}}|f(y)|\,dy.
\end{aligned} \tag{52}$$

It follows that

$$\begin{aligned}
\left|T_{\Omega,\alpha}^A f(x)\right| &\leq \sum_{k=-\infty}^{\infty}\left(C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right. \\
&\quad \left.\cdot \int_{2^{k-1}r\leq|x-y|<2^k r}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}}|f(y)|\,dy\right) \\
&\leq C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
&\quad \cdot \sum_{k=-\infty}^{\infty}\int_{2^{k-1}r\leq|x-y|<2^k r}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}}|f(y)|\,dy \\
&= C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}}|f(y)|\,dy \\
&= C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\overline{T}_{\Omega,\alpha+\beta}f(x).
\end{aligned} \tag{53}$$

Thus, we finish the proof of Theorem 26. □

The following theorem is a key theorem in proving (18) of Theorem 8.

Theorem 27. *Under the same assumptions of Theorem 8, $\overline{T}_{\Omega,\alpha+\beta}$ is bounded from $L^{p,\kappa}(w^p,w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

Proof. Fix a ball $B(x_0,r_B)$, we decompose $f=f_1+f_2$ with $f_1=f\chi_{2B}$. Then we have

$$\begin{aligned}
&\left(\frac{1}{w^q(B)^{\kappa q/p}}\int_B\left|\overline{T}_{\Omega,\alpha+\beta}f(x)\right|^q w^q(x)\,dx\right)^{1/q} \\
&\leq \frac{1}{w^q(B)^{\kappa/p}}\left(\int_B\left|\overline{T}_{\Omega,\alpha+\beta}f_1(x)\right|^q w^q(x)\,dx\right)^{1/q} \\
&\quad + \frac{1}{w^q(B)^{\kappa/p}}\left(\int_B\left|\overline{T}_{\Omega,\alpha+\beta}f_2(x)\right|^q w^q(x)\,dx\right)^{1/q} \\
&\coloneqq J_1+J_2.
\end{aligned} \tag{54}$$

We estimate J_1 at first. By Remark 7(a) we know that $w^q\in\Delta_2$. Then by Theorem 18(a) and the fact that $w^q\in\Delta_2$ we get,

$$\begin{aligned}
J_1 &\leq \frac{1}{w^q(B)^{\kappa/p}}\left\|\overline{T}_{\Omega,\alpha+\beta}f_1\right\|_{L^q(w^q)} \\
&\leq \frac{C}{w^q(B)^{\kappa/p}}\|f_1\|_{L^p(w^p)} \\
&= \frac{C}{w^q(B)^{\kappa/p}}\left(\int_{2B}|f(x)|^p w(x)^p\,dx\right)^{1/p} \\
&\leq C\|f\|_{L^{p,\kappa}(w^p,w^q)}\frac{w^q(2B)^{\kappa/p}}{w^q(B)^{\kappa/p}} \leq C\|f\|_{L^{p,\kappa}(w^p,w^q)}.
\end{aligned} \tag{55}$$

Now we consider the term J_2. By Hölder's inequality, we have

$$\begin{aligned}\left|\overline{T}_{\Omega,\alpha+\beta}f_2(x)\right| &= \sum_{j=1}^{\infty}\int_{2^{j+1}B\backslash 2^jB}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}}|f(y)|\,dy \\ &\le C\sum_{j=1}^{\infty}\left(\int_{2^{j+1}B}|\Omega(x-y)|^s dy\right)^{1/s} \\ &\cdot\left(\int_{2^{j+1}B\backslash 2^jB}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha-\beta)s'}}dy\right)^{1/s'} \\ &:= C\sum_{j=1}^{\infty}\left(I_{1j}I_{2j}\right).\end{aligned} \tag{56}$$

We will estimate I_{1j}, I_{2j}, respectively. Let $z = x - y$; then for $x \in B$, $y \in 2^{j+1}B$, we have $z \in 2^{j+2}B$. Noticing that Ω is homogeneous of degree zero and $\Omega \in L^s(S^{n-1})$, then we have

$$\begin{aligned}I_{1j} &= \left(\int_{2^{j+2}B}|\Omega(z)|^s dz\right)^{1/s} \\ &= \left(\int_0^{2^{j+2}r_B}\int_{S^{n-1}}\left|\Omega(z')\right|^s dz' r^{n-1}dr\right)^{1/s} \\ &= C\|\Omega\|_{L^s(S^{n-1})}\left|2^{j+2}B\right|^{1/s},\end{aligned} \tag{57}$$

where $z' = z/|z|$. For $x \in B$, $y \in (2B)^c$, we have $|x-y| \sim |x_0 - y|$. Thus,

$$I_{2j} \le \frac{C}{\left|2^{j+1}B\right|^{1-(\alpha+\beta)/n}}\left(\int_{2^{j+1}B}|f(y)|^{s'}dy\right)^{1/s'}. \tag{58}$$

By Hölder's inequality and $w^{s'} \in A(p/s', q/s')$, we get

$$\begin{aligned}&\left(\int_{2^{j+1}B}|f(y)|^{s'}dy\right)^{1/s'} \\ &\quad\le C\left(\int_{2^{j+1}B}|f(y)|^p w(y)^p dy\right)^{1/p} \\ &\quad\cdot\left(\int_{2^{j+1}B}w(y)^{-ps'/(p-s')}dy\right)^{(p-s')/ps'} \\ &\quad\le C\|f\|_{L^{p,\kappa}(w^p,w^q)}w^q\left(2^{j+1}B\right)^{\kappa/p} \\ &\quad\cdot\left(\int_{2^{j+1}B}w(y)^{-ps'/(p-s')}dy\right)^{(p-s')/ps'} \\ &\quad\le C\|f\|_{L^{p,\kappa}(w^p,w^q)}w^q\left(2^{j+1}B\right)^{\kappa/p} \\ &\quad\cdot\frac{\left|2^{j+1}B\right|^{(pq-s'q+s'p)/pqs'}}{w^q\left(2^{j+1}B\right)^{1/q}}.\end{aligned} \tag{59}$$

Thus,

$$\begin{aligned}\left|\overline{T}_{\Omega,\alpha+\beta}f_2(x)\right| &\le C\sum_{j=1}^{\infty}\left(I_{1j}I_{2j}\right) \\ &\le C\sum_{j=1}^{\infty}\|f\|_{L^{p,\kappa}(w^p,w^q)}\frac{1}{w^q\left(2^{j+1}B\right)^{1/q-\kappa/p}}.\end{aligned} \tag{60}$$

So we get

$$J_2 \le C\|f\|_{L^{p,\kappa}(w^p,w^q)}\sum_{j=1}^{\infty}\frac{w^q(B)^{1/q-\kappa/p}}{w^q\left(2^{j+1}B\right)^{1/q-\kappa/p}}. \tag{61}$$

We know from Remark 7(a) and Lemma 19 that w^q satisfies inequality (33), so the above series converges since the reverse doubling constant is larger than one. Hence,

$$J_2 \le C\|f\|_{L^{p,\kappa}(w^p,w^q)}. \tag{62}$$

Therefore, the proof of Theorem 27 is completed. □

Remark 28. It is worth noting that Theorem 27 is essentially verifying the multilinear fractional operator $T_{\Omega,\alpha}$ is bounded on weighted Morrey spaces.

Now we are in the position of proving Theorem 8.

We will obtain (18) immediately in combination of Theorems 26 and 27.

Then let us turn to prove (19).

Set

$$\begin{aligned}&\overline{T}^A_{\Omega,\alpha}f(x) \\ &\quad= \int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}}\left|R_m(A;x,y)\right||f(y)|\,dy, \\ &\qquad 0 \le \alpha < n,\end{aligned} \tag{63}$$

where $\Omega \in L^s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero in $\mathbb{R}^n$. It is easy to see inequality (18) also holds for $\overline{T}^A_{\Omega,\alpha}$. On the other hand, for any $r > 0$, we have

$$\begin{aligned}&\overline{T}^A_{\Omega,\alpha}f(x) \\ &\quad\ge \int_{|x-y|<r}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}}\left|R_m(A;x,y)\right||f(y)|\,dy \\ &\quad\ge \frac{1}{r^{n-\alpha+m-1}} \\ &\quad\cdot\int_{|x-y|<r}|\Omega(x-y)|\left|R_m(A;x,y)\right||f(y)|\,dy.\end{aligned} \tag{64}$$

Taking the supremum for $r > 0$ on the inequality above, we get

$$\overline{T}^A_{\Omega,\alpha}f(x) \ge M^A_{\Omega,\alpha}f(x). \tag{65}$$

Thus, we can immediately obtain (19) from (65) and (18).

Similarly as before, we give the following theorem at first before proving Theorem 9, since it plays an important role in the proof of Theorem 9. Set

$$\overline{T}_{\Omega,\beta} f(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\beta}} |f(y)| \, dy, \qquad (66)$$

where $\Omega \in L^s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero in $\mathbb{R}^n$.

Theorem 29. *Under the assumptions of Theorem 9, $\overline{T}_{\Omega,\beta}$ is bounded from $L^{p,\kappa}(w^p, w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

The proof of Theorem 29 can be treated as that of Theorem 27 with only slight modifications; we omit its proof here.

Now, let us prove Theorem 9. It is not difficult to see that (20) can be easily obtained from Theorems 26 and 29. Then we can immediately arrive at (21) from (65) and (20).

From now on, we are in the place of showing Theorem 10. We prove (22) at first. Fixing any cube Q with center at x and diameter r, denote $\overline{Q} = 2Q$ and set

$$A_{\overline{Q}}(y) = A(y) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} m_{\overline{Q}}(D^\gamma A) \, y^\gamma. \qquad (67)$$

Noticing that equality (67) is the special case of equality (44) when $k = 1$. Thus, equalities (45) and (46) also hold for $A_{\overline{Q}}(y)$. We decompose f as $f = f\chi_{\overline{Q}} + f\chi_{(\overline{Q})^c} := f_1 + f_2$. Then we have

$$\begin{aligned}
&\frac{1}{w^q(Q)^{\kappa/p}} \left(\int_Q \left| T^A_{\Omega,\alpha} f(y) \right|^q w(y)^q \, dy \right)^{1/q} \\
&\quad \le \frac{1}{w^q(Q)^{\kappa/p}} \left(\int_Q \left| T^A_{\Omega,\alpha} f_1(y) \right|^q w(y)^q \, dy \right)^{1/q} \\
&\quad\quad + \frac{1}{w^q(Q)^{\kappa/p}} \left(\int_Q \left| T^A_{\Omega,\alpha} f_2(y) \right|^q w(y)^q \, dy \right)^{1/q} \\
&\quad := I + II.
\end{aligned} \qquad (68)$$

By Theorem 20(a) and Remark 7(a) that $w^q \in \Delta_2$, we have

$$\begin{aligned}
I &\le \frac{C}{w^q(Q)^{\kappa/p}} \\
&\quad \cdot \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \left(\int_{\overline{Q}} |f(y)|^p w(y)^p \, dy \right)^{1/p} \\
&= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p,w^q)} \left(\frac{w^q(\overline{Q})}{w^q(Q)} \right)^{\kappa/p} \\
&\le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p,w^q)}.
\end{aligned} \qquad (69)$$

Next, we consider the term $T^A_{\Omega,\alpha} f_2(y)$ contained in II. By Lemma 15 and equality (45), (46), we have

$$\begin{aligned}
&\left| T^A_{\Omega,\alpha} f_2(y) \right| \\
&\quad \le \int_{(\overline{Q})^c} \frac{\left| R_m(A_{\overline{Q}}; y, z) \right|}{|y-z|^{m-1}} \frac{|\Omega(y-z)|}{|y-z|^{n-\alpha}} |f(z)| \, dz \\
&\quad \le C \int_{(\overline{Q})^c} \sum_{|\gamma|=m-1} \left(\frac{1}{|I^z_y|} \int_{I^z_y} \left| D^\gamma A_{\overline{Q}}(t) \right|^l dt \right)^{1/l} \\
&\quad\quad \cdot |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz \\
&\quad\quad + C \int_{(\overline{Q})^c} \sum_{|\gamma|=m-1} \left| D^\gamma A(z) - m_{\overline{Q}}(D^\gamma A) \right| |\Omega(y-z)| \\
&\quad\quad \cdot \frac{|f(z)|}{|y-z|^{n-\alpha}} dz := II_1 + II_2.
\end{aligned} \qquad (70)$$

We estimate II_1 and II_2, respectively. By Lemma 21(a) and (b), Hölder's inequality, and $w^{s'} \in A(p/s', q/s')$, we get

$$\begin{aligned}
II_1 &\le C \sum_{|\gamma|=m-1} \left(\frac{1}{|I^z_y|} \int_{I^z_y} \left| D^\gamma A(t) - m_{\overline{Q}}(D^\gamma A) \right|^l dt \right)^{1/l} \\
&\quad \cdot \sum_{j=1}^\infty \int_{2^{j+1}Q \setminus 2^j Q} |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz \\
&\le C \sum_{|\gamma|=m-1} \Bigg[\Bigg(\frac{1}{|I^z_y|} \\
&\quad \cdot \int_{I^z_y} \left| D^\gamma A(t) - m_{I^z_y}(D^\gamma A) \right|^l dt \Bigg)^{1/l} + \Big| m_{I^z_y}(D^\gamma A) \\
&\quad - m_{5n\overline{Q}}(D^\gamma A) \Big| + \left| m_{5n\overline{Q}}(D^\gamma A) - m_{\overline{Q}}(D^\gamma A) \right| \Bigg] \\
&\quad \cdot \sum_{j=1}^\infty \left(\int_{2^{j+1}Q} |\Omega(y-z)|^s dz \right)^{1/s} \\
&\quad \cdot \left(\int_{2^{j+1}Q \setminus 2^j Q} \frac{|f(z)|^{s'}}{|y-z|^{(n-\alpha)s'}} dz \right)^{1/s'} \\
&\le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|\Omega\|_{L^s(S^{n-1})} \\
&\quad \cdot \sum_{j=1}^\infty \frac{|2^{j+2}Q|^{1/s}}{|2^j Q|^{1-\alpha/n}} \left(\int_{2^{j+1}Q} |f(z)|^{s'} dz \right)^{1/s'}
\end{aligned}$$

$$\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\alpha/n}} \left(\int_{2^{j+1}Q} |f(z)|^p \cdot w(z)^p \, dz \right)^{1/p} \cdot \left(\int_{2^{j+1}Q} w(z)^{-ps'/(p-s')} \, dz \right)^{(p-s')/ps'} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p,w^q)} \cdot \sum_{j=1}^{\infty} \frac{1}{w^q (2^{j+1}Q)^{1/q-\kappa/p}}. \tag{71}$$

For $y \in Q$, $z \in (\overline{Q})^c$, we have $|y-z| \sim |x-z|$, so we obtain

$$II_2 \leq C \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^jQ} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)| \cdot |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz + C \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^jQ} \sum_{|\gamma|=m-1} |m_{2^{j+1}Q}(D^\gamma A) - m_{\overline{Q}}(D^\gamma A)| \cdot |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz := II_{21} + II_{22}. \tag{72}$$

By Hölder's inequality, we get

$$\begin{aligned} II_{21} &\leq C \sum_{j=1}^{\infty} \frac{1}{|2^jQ|^{1-\frac{\alpha}{n}}} \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)| |\Omega(y-z)| |f(z)| \, dz \\ &\leq C \sum_{j=1}^{\infty} \frac{1}{|2^jQ|^{1-\frac{\alpha}{n}}} \left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{s'} |f(z)|^{s'} dz \right)^{1/s'} \left(\int_{2^{j+1}Q} |\Omega(y-z)|^s dz \right)^{1/s} \\ &\leq C \|\Omega\|_{L^s(S^{n-1})} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\frac{\alpha}{n}}} \left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{s'} |f(z)|^{s'} dz \right)^{1/s'} \\ &\leq C \|\Omega\|_{L^s(S^{n-1})} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\frac{\alpha}{n}}} \left(\int_{2^{j+1}Q} |f(z)|^p w(z)^p dz \right)^{1/p} \\ &\quad \cdot \left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{ps'/(p-s')} w(z)^{-ps'/(p-s')} dz \right)^{(p-s')/ps'}. \end{aligned} \tag{73}$$

We estimate the part containing the function $D^\gamma A$ as follows:

$$\begin{aligned} &\left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{ps'/(p-s')} w(z)^{-ps'/(p-s')} dz \right)^{(p-s')/ps'} \\ &\leq C \left(\left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q, w^{-ps'/(p-s')}}(D^\gamma A)|^{ps'/(p-s')} w(z)^{-ps'/(p-s')} dz \right)^{(p-s')/ps'} \right. \\ &\quad \left. + \sum_{|\gamma|=m-1} |m_{2^{j+1}Q, w^{-ps'/(p-s')}}(D^\gamma A) - m_{2^{j+1}Q}(D^\gamma A)| \, w^{-ps'/(p-s')} (2^{j+1}Q)^{(p-s')/ps'} \right) := III + IV. \end{aligned} \tag{74}$$

For the term III, since $w^{s'} \in A(p/s', q/s')$, we then have $w^{-ps'/(p-s')} \in A_{t'} \subset A_\infty$ by Remark 7(b). Thus, by Lemma 25 that the norm of $\mathrm{BMO}(w^{-ps'/(p-s')})$ is equivalent to the norm of $\mathrm{BMO}(\mathbb{R}^n)$ and $w^{s'} \in A(p/s', q/s')$, we have

$$
\begin{aligned}
III &\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} w^{-ps'/(p-s')}\left(2^{j+1}Q\right)^{(p-s')/ps'} \\
&= C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \frac{\left|2^{j+1}Q\right|^{(p-s')/ps'+1/q}}{w^{q}\left(2^{j+1}Q\right)^{1/q}}.
\end{aligned} \tag{75}
$$

For the term IV, by Lemma 21(a), there exist $C_1, C_2 > 0$ such that for any cube Q and $s > 0$,

$$
\begin{aligned}
&\left|\left\{t \in 2^{j+1}Q : \sum_{|\gamma|=m-1} \left|D^{\gamma}A(t) - m_{2^{j+1}Q}(D^{\gamma}A)\right| > s\right\}\right| \\
&\quad \leq C_1 \left|2^{j+1}Q\right| e^{-C_2 s/(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{*})},
\end{aligned} \tag{76}
$$

since $\sum_{|\gamma|=m-1}(D^{\gamma}A) \in \mathrm{BMO}$. Then by Lemma 24(2), we have

$$
\begin{aligned}
&w\left(\left\{t \in 2^{j+1}Q : \sum_{|\gamma|=m-1} \left|D^{\gamma}A(t) - m_{2^{j+1}Q}(D^{\gamma}A)\right|\right.\right. \\
&\quad \left.\left. > s\right\}\right) \leq Cw\left(2^{j+1}Q\right) e^{-C_2 s\delta/(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{*})},
\end{aligned} \tag{77}
$$

for some $\delta > 0$. Hence it implies

$$
\begin{aligned}
&\sum_{|\gamma|=m-1} \left|m_{2^{j+1}Q, w^{-ps'/(p-s')}}(D^{\gamma}A) - m_{2^{j+1}Q}(D^{\gamma}A)\right| \\
&\quad \leq \frac{1}{w^{-ps'/(p-s')}\left(2^{j+1}Q\right)} \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} \left|D^{\gamma}A(t)\right. \\
&\quad \left. - m_{2^{j+1}Q}(D^{\gamma}A)\right| w^{-ps'/(p-s')}(t)\,dt \\
&\quad = \frac{C}{w^{-ps'/(p-s')}\left(2^{j+1}Q\right)} \int_{0}^{\infty} w^{-ps'/(p-s')}\left(\left\{t\right.\right. \\
&\quad \in 2^{j+1}Q : \sum_{|\gamma|=m-1} \left|D^{\gamma}A(t) - m_{2^{j+1}Q}(D^{\gamma}A)\right| \\
&\quad \left.\left. > s\right\}\right) ds \leq \frac{C}{w^{-ps'/(p-s')}\left(2^{j+1}Q\right)} \\
&\quad \cdot \int_{0}^{\infty} w^{-ps'/(p-s')}\left(2^{j+1}Q\right) e^{-C_2 s\delta/(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{*})} ds \\
&\quad = C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*}.
\end{aligned} \tag{78}
$$

As a result,

$$
\begin{aligned}
IV &\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} w^{-ps'/(p-s')}\left(2^{j+1}Q\right)^{(p-s')/ps'} \\
&= C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \frac{\left|2^{j+1}Q\right|^{(p-s')/ps'+1/q}}{w^{q}\left(2^{j+1}Q\right)^{1/q}}.
\end{aligned} \tag{79}
$$

Thus,

$$
\begin{aligned}
II_{21} &\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \|f\|_{L^{p,\kappa}(w^p,w^q)} \\
&\quad \cdot \sum_{j=1}^{\infty} \frac{1}{w^{q}\left(2^{j+1}Q\right)^{1/q-\kappa/p}}.
\end{aligned} \tag{80}
$$

For the term II_{22}, by Lemma 21(c), Hölder's inequality, and $w^{s'} \in A(p/s', q/s')$, we get

$$
\begin{aligned}
II_{22} &\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \\
&\quad \cdot \sum_{j=1}^{\infty} j \int_{2^{j+1}Q\setminus 2^{j}Q} \left|\Omega(y-z)\right| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz \\
&\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \|\Omega\|_{L^{s}(S^{n-1})} \sum_{j=1}^{\infty} j \\
&\quad \cdot \frac{\left|2^{j+2}Q\right|^{1/s}}{\left|2^{j}Q\right|^{1-\alpha/n}} \left(\int_{2^{j+1}Q} |f(z)|^{s'} dz\right)^{1/s'} \\
&\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \sum_{j=1}^{\infty} j \\
&\quad \cdot \frac{\left|2^{j+2}Q\right|^{1/s}}{\left|2^{j}Q\right|^{1-\alpha/n}} \left(\int_{2^{j+1}Q} |f(z)|^{p} w(z)^{p} dz\right)^{1/p} \\
&\quad \cdot \left(\int_{2^{j+1}Q} w(z)^{-ps'/(p-s')} dz\right)^{(p-s')/ps'} \\
&\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \|f\|_{L^{p,\kappa}(w^p,w^q)} \\
&\quad \cdot \sum_{j=1}^{\infty} \frac{j}{w^{q}\left(2^{j+1}Q\right)^{1/q-\kappa/p}}.
\end{aligned} \tag{81}
$$

Hence,

$$
\begin{aligned}
\left|T_{\Omega,\alpha}^{A} f_2(y)\right| &\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \|f\|_{L^{p,\kappa}(w^p,w^q)} \\
&\quad \cdot \sum_{j=1}^{\infty} \frac{j}{w^{q}\left(2^{j+1}Q\right)^{1/q-\kappa/p}}.
\end{aligned} \tag{82}
$$

Therefore,

$$
\begin{aligned}
II &\leq C \sum_{|\gamma|=m-1} \left\|D^{\gamma}A\right\|_{*} \|f\|_{L^{p,\kappa}(w^p,w^q)} \\
&\quad \cdot \sum_{j=1}^{\infty} j \frac{w^{q}(Q)^{1/q-\kappa/p}}{w^{q}\left(2^{j+1}Q\right)^{1/q-\kappa/p}}
\end{aligned}
$$

$$\begin{aligned}&\le C\sum_{|\gamma|=m-1}\left\|D^{\gamma}A\right\|_{*}\|f\|_{L^{p,\kappa}(w^p,w^q)}\sum_{j=1}^{\infty}\frac{j}{\left(D_1^{j+1}\right)^{1/q-\kappa/p}}\\&\le C\sum_{|\gamma|=m-1}\left\|D^{\gamma}A\right\|_{*}\|f\|_{L^{p,\kappa}(w^p,w^q)},\end{aligned}\tag{83}$$

where $D_1>1$ is the reverse doubling constant. Consequently,

$$\begin{aligned}&\frac{1}{w^q(Q)^{\kappa/p}}\left(\int_Q\left|T^A_{\Omega,\alpha}f(y)\right|^q w(y)^q\,dy\right)^{1/q}\\&\qquad\le C\sum_{|\gamma|=m-1}\left\|D^{\gamma}A\right\|_{*}\|f\|_{L^{p,\kappa}(w^p,w^q)}.\end{aligned}\tag{84}$$

Taking supremum over all cubes in $\mathbb{R}^n$ on both sides of the above inequality, we complete the proof of (22) of Theorem 10.

It is not difficult to see that inequality (23) is easy to get from (22) and (65).

Proof of Theorem 11. We consider (25) firstly. Let Q be the same as in the proof of (22) and denote $\overline{Q}=2Q$; we decompose f as $f=f\chi_{\overline{Q}}+f\chi_{(\overline{Q})^c}:=f_1+f_2$. Then we have

$$\begin{aligned}&\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q\left|[A,T_\Omega]f(y)\right|^p w(y)\,dy\right)^{1/p}\\&\quad\le\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q\left|[A,T_\Omega]f_1(y)\right|^p w(y)\,dy\right)^{1/p}\\&\quad+\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q\left|[A,T_\Omega]f_2(y)\right|^p w(y)\,dy\right)^{1/p}\\&\quad:=I+II.\end{aligned}\tag{85}$$

By Theorem 22(a) and Lemma 24(1) that $w\in\Delta_2$, we get

$$\begin{aligned}I&\le\frac{1}{w(Q)^{\kappa/p}}\left\|[A,T_\Omega]f_1\right\|_{L^p(w)}\\&\le\frac{C}{w(Q)^{\kappa/p}}\|A\|_{*}\|f_1\|_{L^p(w)}\\&=C\|A\|_{*}\|f\|_{L^{p,\kappa}(w)}\frac{w\left(\overline{Q}\right)^{\kappa/p}}{w(Q)^{\kappa/p}}\le C\|A\|_{*}\|f\|_{L^{p,\kappa}(w)}.\end{aligned}\tag{86}$$

For $|[A,T_\Omega]f_2(y)|$, by Hölder's inequality, we obtain

$$\begin{aligned}&\left|[A,T_\Omega]f_2(y)\right|\le\sum_{j=1}^{\infty}\int_{2^{j+1}Q\setminus 2^jQ}\frac{|\Omega(y-z)|}{|y-z|^n}|A(y)-A(z)|\\&\quad\cdot|f(z)|\,dz\\&\quad\le C\sum_{j=1}^{\infty}\frac{1}{|2^jQ|}\left(\int_{2^{j+1}Q}|\Omega(y-z)|^s\,dz\right)^{1/s}\\&\quad\cdot\left(\int_{2^{j+1}Q}|A(y)-A(z)|^{s'}|f(z)|^{s'}\,dz\right)^{1/s'}\le C\|\Omega\|_{L^s}\\&\quad\cdot\sum_{j=1}^{\infty}\frac{\left|2^{j+2}Q\right|^{1/s}}{|2^jQ|}\left|A(y)-m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)\right|\\&\quad\cdot\left(\int_{2^{j+1}Q}|f(z)|^{s'}\,dz\right)^{1/s'}+C\|\Omega\|_{L^s}\\&\quad\cdot\sum_{j=1}^{\infty}\frac{\left|2^{j+2}Q\right|^{1/s}}{|2^jQ|}\left(\int_{2^{j+1}Q}\left|m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)-A(z)\right|^{s'}\right.\\&\quad\left.\cdot|f(z)|^{s'}\,dz\right)^{1/s'}:=II_1(y)+II_2.\end{aligned}\tag{87}$$

Next we estimate $II_1(y)$ and II_2, respectively. By Hölder's inequality and $w\in A_{p/s'}$, we have

$$\begin{aligned}&\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q II_1(y)^p w(y)\,dy\right)^{1/p}\\&=C\frac{\|\Omega\|_{L^s}}{w(Q)^{\kappa/p}}\sum_{j=1}^{\infty}\frac{\left|2^{j+2}Q\right|^{1/s}}{|2^jQ|}\left(\int_Q\left|A(y)-m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)\right|^p\left(\int_{2^{j+1}Q}|f(z)|^{s'}\,dz\right)^{p/s'}w(y)\,dy\right)^{1/p}\\&\le C\frac{\|f\|_{L^{p,\kappa}(w)}}{w(Q)^{\frac{\kappa}{p}}}\sum_{j=1}^{\infty}\frac{\left|2^{j+2}Q\right|^{1/s}}{|2^jQ|}w\left(2^{j+1}Q\right)^{\kappa/p}w^{-s'/(p-s')}\left(2^{j+1}Q\right)^{(p-s')/ps'}\cdot\left(\int_Q\left|A(y)-m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)\right|^p w(y)\,dy\right)^{1/p}\\&\le C\frac{\|f\|_{L^{p,\kappa}(w)}}{w(Q)^{\kappa/p}}\sum_{j=1}^{\infty}\frac{1}{w\left(2^{j+1}Q\right)^{(1-\kappa)/p}}\left(\int_Q\left|A(y)-m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)\right|^p w(y)\,dy\right)^{1/p}.\end{aligned}\tag{88}$$

We estimate the part containing $m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)$ as follows:

$$\begin{aligned}
&\left(\int_Q \left|A(y)-m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)\right|^p w(y)\,dy\right)^{1/p} \\
&\quad\leq \left(\int_Q \left|A(y)-m_{Q,w}(A)\right|^p w(y)\,dy\right)^{1/p} \\
&\qquad+\left|m_{Q,w}(A)-m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)\right| w(Q)^{1/p} \\
&\quad\coloneqq III+IV.
\end{aligned} \tag{89}$$

For the term III, notice that $w\in A_{p/s'}\subset A_\infty$; we thus get by Lemma 25 that

$$III\leq \|A\|_* \, w(Q)^{1/p}. \tag{90}$$

Next we estimate IV. By Lemmas 21(c) and 25, we have

$$\begin{aligned}
&\left|m_{Q,w}(A)-m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)\right| \\
&\quad\leq \left|m_{Q,w}(A)-m_Q(A)\right|+\left|m_Q(A)-m_{2^{j+1}Q}(A)\right| \\
&\qquad+\left|m_{2^{j+1}Q}(A)-m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)\right|\leq \frac{1}{w(Q)} \\
&\qquad\cdot\int_Q \left|A(t)-m_Q(A)\right| w(t)\,dt+2^n(j+1)\|A\|_* \\
&\qquad+\frac{1}{w^{-s'/(p-s')}(2^{j+1}Q)} \\
&\qquad\cdot\int_{2^{j+1}Q}\left|A(t)-m_{2^{j+1}Q}(A)\right| w^{-s'/(p-s')}(t)\,dt \\
&\quad\leq C(j+1)\|A\|_*.
\end{aligned} \tag{91}$$

Hence,

$$IV\leq C(j+1)\|A\|_* \, w(Q)^{1/p}. \tag{92}$$

As a result,

$$\begin{aligned}
&\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q II_1(y)^p w(y)\,dy\right)^{1/p} \\
&\quad\leq C\|A\|_* \|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}(j+1)\frac{w(Q)^{(1-\kappa)/p}}{w(2^{j+1}Q)^{(1-\kappa)/p}} \\
&\quad\leq C\|A\|_*\|f\|_{L^{p,\kappa}(w)}.
\end{aligned} \tag{93}$$

For II_2, by Hölder's inequality and $w\in A_{p/s'}$, we get

$$\begin{aligned}
II_2 &\leq C\|\Omega\|_{L^s} \\
&\cdot\sum_{j=1}^{\infty}\frac{\left|2^{j+2}Q\right|^{1/s}}{\left|2^jQ\right|}\left(\int_{2^{j+1}Q}|f(z)|^p w(z)\,dz\right)^{1/p} \\
&\cdot\left(\int_{2^{j+1}Q}\left|m_{2^{j+1}Q,w^{-s'/(p-s')}}(A)-A(z)\right|^{ps'/(p-s')}\right. \\
&\left.\cdot\, w(z)^{-s'/(p-s')}\,dz\right)^{(p-s')/ps'}\leq C\|\Omega\|_{L^s}\|A\|_* \\
&\cdot\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}\frac{\left|2^{j+2}Q\right|^{1/s}}{\left|2^jQ\right|} w\left(2^{j+1}Q\right)^{\kappa/p} \\
&\cdot\, w^{-s'/(p-s')}\left(2^{j+1}Q\right)^{(p-s')/ps'}\leq C\|A\|_*\|f\|_{L^{p,\kappa}(w)} \\
&\cdot\sum_{j=1}^{\infty}\frac{1}{w(2^{j+1}Q)^{(1-\kappa)/p}}.
\end{aligned} \tag{94}$$

Therefore,

$$\begin{aligned}
&\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q II_2^p w(y)\,dy\right)^{1/p} \\
&\quad\leq C\|A\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{j=1}^{\infty}\frac{w(Q)^{(1-\kappa)/p}}{w(2^{j+1}Q)^{(1-\kappa)/p}} \\
&\quad\leq C\|A\|_*\|f\|_{L^{p,\kappa}(w)}.
\end{aligned} \tag{95}$$

So far, we have completed the proof of (25). □

Inequality (26) can be immediately obtained from (65) and (25).

Proof of Theorem 12. As before, we prove (27) at first. Assume Q to be the same as in the proof of (22), denote $\overline{Q}=2Q$, and set

$$A_{\overline{Q}}(y)=A(y)-m_{\overline{Q}}(\nabla A)\,y. \tag{96}$$

We also decompose f according to $\overline{Q}$: $f=f\chi_{\overline{Q}}+f\chi_{(\overline{Q})^c}\coloneqq f_1+f_2$. Then we get

$$\begin{aligned}
&\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q\left|\widetilde{T}_\Omega^A f(y)\right|^p w(y)\,dy\right)^{1/p} \\
&\quad\leq\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q\left|\widetilde{T}_\Omega^A f_1(y)\right|^p w(y)\,dy\right)^{1/p} \\
&\qquad+\frac{1}{w(Q)^{\kappa/p}}\left(\int_Q\left|\widetilde{T}_\Omega^A f_2(y)\right|^p w(y)\,dy\right)^{1/p} \\
&\quad\coloneqq I+II.
\end{aligned} \tag{97}$$

For I, Theorem 23 and Lemma 24(1) imply

$$\begin{aligned} I &\leq \frac{1}{w(Q)^{\kappa/p}} \left\| \widetilde{T}_{\Omega}^{A} f_1 \right\|_{L^p(w)} \\ &\leq \frac{C}{w(Q)^{\kappa/p}} \|\Omega\|_{\infty} \|\nabla A\|_{*} \|f_1\|_{L^p(w)} \\ &\leq C \|\nabla A\|_{*} \|f\|_{L^{p,\kappa}(w)} \frac{w(\overline{Q})^{\kappa/p}}{w(Q)^{\kappa/p}} \\ &\leq C \|\nabla A\|_{*} \|f\|_{L^{p,\kappa}(w)} . \end{aligned} \tag{98}$$

We will omit the proof for II since it is similar to and even easier than the part of II in the proof of (22), except that we use the conditions $w \in A_p$, $\Omega \in L^{\infty}(S^{n-1})$, $m = 2$, and $f \in L^{p,\kappa}(w)$. For inequality (28), it can be easily proved by (27) and (65). Thus, we complete the proof of Theorem 12. □

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Uniform Attractors for Nonclassical Diffusion Equations with Memory

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We introduce a new method (or technique), asymptotic contractive method, to verify uniform asymptotic compactness of a family of processes. After that, the existence and the structure of a compact uniform attractor for the nonautonomous nonclassical diffusion equation with fading memory are proved under the following conditions: the nonlinearity f satisfies the polynomial growth of arbitrary order and the time-dependent forcing term g is only translation-bounded in $L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$.

1. Introduction

In this paper, we consider dynamical behavior of solutions for the following nonclassical diffusion equation with a fading memory term:

$$\begin{aligned} &u_t(t) - \Delta u_t(t) - \Delta u(t) - \int_0^\infty k(s)\,\Delta u(t-s)\,ds \\ &\quad + f(u(t)) = g(t), \quad \text{in } \Omega \times (\tau, \infty). \end{aligned} \tag{1}$$

The problem is supplemented with the boundary condition,

$$u(x,t)|_{\partial\Omega} = 0, \quad \forall t \in \mathbb{R}, \tag{2}$$

and initial condition,

$$u(x,t) = u_\tau(x,t), \quad t \leqslant \tau, \tag{3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ and $g(t)$ is a given external time-dependent forcing.

Concerning the memory kernel $\mu(s) = -k'(s)$, as in [1–6], we assume the following hypotheses:

$$\begin{aligned} &\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \\ &\qquad \mu(s) \geqslant 0,\ \mu'(s) \leqslant 0\ \forall s \in \mathbb{R}^+, \end{aligned} \tag{4}$$

and there is a constant $\delta > 0$, such that

$$\mu'(s) + \delta\mu(s) \leqslant 0 \quad \forall s \in \mathbb{R}^+. \tag{5}$$

From (4) and (5), we get

$$\mu(\infty) = \lim_{s\to+\infty} \mu(s) = 0. \tag{6}$$

We also denote

$$m_0 := \int_0^\infty \mu(s)\,ds < \infty. \tag{7}$$

As Wang and Zhong [5], we introduce the past history of u, that is,

$$\eta^t = \eta^t(x,s) := \int_0^s u(x,t-r)\,dr, \quad s \in \mathbb{R}^+, \tag{8}$$

as a new variable of the system, which will be ruled by a supplementary equation. Denote

$$\begin{aligned} \eta_t^t &= \frac{\partial}{\partial t}\eta^t, \\ \eta_s^t &= \frac{\partial}{\partial s}\eta^t, \end{aligned} \tag{9}$$

and (1) transforms into the following system:

$$\begin{aligned} &u_t(t) - \Delta u_t(t) - \Delta u(t) - \int_0^\infty \mu(s)\,\Delta\eta^t(s)\,ds \\ &\quad + f(u(t)) = g(t), \\ &\eta_t^t = -\eta_s^t + u, \end{aligned} \tag{10}$$

with initial-boundary conditions

$$
\begin{gathered}
u(x,t)|_{\partial\Omega} = 0, \\
\eta^t(x,s)\big|_{\partial\Omega\times\mathbb{R}^+} = 0, \\
t \geq \tau, \\
u(x,\tau) = u_\tau(x,\tau), \quad x \in \Omega, \\
\eta^\tau = \eta^\tau(x,s) = \int_0^s u_\tau(x,\tau - r)\,dr, \\
(x,s) \in \Omega \times \mathbb{R}^+.
\end{gathered} \tag{11}
$$

The past history $u_\tau(\tau - s)$ of the variable u satisfies the condition as follows: there exist positive constants $\mathfrak{R}$ and $\varrho \leq \delta$ (from (5)), such that

$$\int_0^\infty e^{-\varrho s}\left\|u_\tau(\tau - s)\right\|_0^2 ds \leq \mathfrak{R}. \tag{12}$$

The nonlinearity $f \in \mathscr{C}^1$ fulfills $f(0) = 0$, along with arbitrary order polynomial growth restriction

$$\gamma_1|s|^p - \beta_1 \leq f(s)s \leq \gamma_2|s|^p + \beta_2, \quad p \geq 2, \tag{13}$$

and the dissipation condition

$$f'(s) \geq -l, \tag{14}$$

where γ_i, β_i $(i = 1, 2)$ and l are positive constants.

This equation appears as an extension of the usual nonclassical diffusion equation in fluid mechanics, solid mechanics, and heat conduction theory (see [7–10]). Equation (1) with a one-time derivative appearing in the highest order term is called pseudoparabolic or Sobolev-Galpern equations [11–13]. Aifantis [7] proposed a general frame for establishing the equations. For certain classes of materials such as polymer and high-viscosity liquids, the diffusive process is nontrivially influenced by the past history of u, which is represented in (1) by the convolution term against a suitable memory kernel characterizing the diffusive species [14].

Proving the existence of uniform attractors for (1) may be a hard task, mainly due to the fact that the nonlinearity satisfies arbitrary polynomial growth condition instead of critical, so Sobolev compact embedding theorems are no longer useful. The asymptotic compactness of solutions cannot be obtained by the usual method (used, e.g., in [2, 5, 6, 15, 16]).

When μ is a Dirac measure at some fixed time instant or when it vanishes, (1) reduces to the usual nonclassical diffusion equation, which has been investigated extensively by many authors, especially about the asymptotic behavior of solutions; see, for example, [16–22] and the references therein. In [21], the author has proved the existence of global attractor in $H_0^1(\Omega)$, when $g(x) \in L^2(\Omega)$ under the assumptions that f satisfies critical exponent growth condition corresponding to $N = 3$ and some additional condition for nonlinearity, which essentially requires that the nonlinearity is subcritical. Recently, the authors in [23] obtained the existence of a global attractor for $g(x) \in H^{-1}(\Omega)$ only under critical nonlinearity, and when $g(x,t) \in L_b^2(\mathbb{R}, L^2(\Omega))$, they proved the asymptotic regularity and the existence of (nonautonomous) exponential attractor. The asymptotic behavior of solutions of this equation has received considerably less attention in the literature under the assumption that the nonlinearity satisfies arbitrary polynomial growth condition. In the ordinary case for some recent results on this equations the reader can refer to Sun et al. [24] and Anh and Toan [25, 26]. Hereafter in [23] the authors mention that these are some mistakes in the coauthor's earlier paper [24]. In [25], they proved that the nonautonomous dynamical system generated by this class of solutions has a pullback attractor. In [27], they proved the existence of global attractor in $H^1(\mathbb{R}^N)$ with the initial data $u_0 \in H^2(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$.

For convenience, hereafter let $|u|$ be the modular (or absolute value) of u and let $|\cdot|_p$ be the norm of $L^p(\Omega)$ $(p \geq 1)$. Let $\|\cdot\|_0 = |\nabla\cdot|_2$ be the norm of $\mathscr{V}$. Denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ and let $\|\cdot\|_{H^{-1}}$ be the norm of $H^{-1}(\Omega)$.

C denotes any positive constant which may be different from line to line even in the same line.

Let $L_\mu^2(\mathbb{R}^+;\mathscr{V})$ be the Hilbert spaces of functions $\varphi : \mathbb{R}^+ \to \mathscr{V}$, endowed with the inner product and norm, respectively. Consider

$$
\begin{gathered}
\langle\varphi,\psi\rangle_{\mu,\mathscr{V}} = \int_0^\infty \mu(s)\langle\varphi(s),\psi(s)\rangle_{\mathscr{V}}\,ds, \\
\|\varphi\|_{\mu,\mathscr{V}}^2 = \int_0^\infty \mu(s)\|\varphi(s)\|_0^2\,ds.
\end{gathered} \tag{15}
$$

We also define the product space $\mathscr{M}_1$:

$$\mathscr{M}_1 = \mathscr{V} \times L_\mu^2(\mathbb{R}^+;\mathscr{V}), \tag{16}$$

endowed with the norm

$$\|z\|_{\mathscr{M}_1}^2 = \left\|(u,\eta^t)\right\|_{\mathscr{M}_1}^2 = \frac{1}{2}\left(|u|_2^2 + \|u\|_0^2 + \left\|\eta^t\right\|_{\mu,\mathscr{V}}^2\right). \tag{17}$$

For convenience, we first show the preliminary result as follows.

Lemma 1 (see [5]). *Let $I = [0,T]$, $\forall T > 0$ and let the memory kernel $\mu(s)$ satisfy (4) and (5). Then for any $\eta^t \in C(I; L_\mu^2(\mathbb{R}^+;\mathscr{V}))$ the following estimate*

$$\left\langle\eta^t,\eta_s^t\right\rangle \geq \frac{\delta}{2}\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2 \tag{18}$$

holds, where δ is from (5).

For the time-dependent forcing $g(x,t)$, we assume the following hypotheses:

$$g \in L_{w,\mathrm{loc}}^2\left(\mathbb{R}; L^2(\Omega)\right). \tag{19}$$

Here $L_{w,\mathrm{loc}}^2(\mathbb{R}; L^2(\Omega))$ is the space $L_{\mathrm{loc}}^2(\mathbb{R}; L^2(\Omega))$ under the local weak convergence topology. Recall that a sequence $\{g_n\}$

converges to g as $n \to \infty$ in $L^2_{w,\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ if and only if

$$\lim_{n\to\infty} \int_{t_1}^{t_2} \int_\Omega v(x,s)\left(g_n(x,s) - g(x,s)\right) ds = 0 \tag{20}$$

for all $[t_1, t_2] \subset \mathbb{R}$ and every $v \in L^2(t_1, t_2; L^2(\Omega))$ holds.

Let $g_0 \in L^2_b(\mathbb{R}; L^2(\Omega))$ (translation-bounded in $L^2_{w,\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$), and introduce a set of functions obtained by time-translation in g_0:

$$\Sigma_0 = \left\{g_0(x, t+h) : h \in \mathbb{R}\right\}. \tag{21}$$

We define the hull of g_0, denoted as $\Sigma = \mathscr{H}(g_0)$, as the closure of Σ_0 with respect to the local weak convergence topology of $L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$. If $g \in \Sigma = \mathscr{H}(g_0)$, then $g \in L^2_b(\mathbb{R}; L^2(\Omega))$; that is,

$$\|g\|^2_{L^2_b} = \sup_{t\in\mathbb{R}} \int_t^{t+1} |g(s)|_2^2 \, ds < \infty, \tag{22}$$

where $\|\cdot\|_{L^2_b}$ means the norm of $L^2_b(\mathbb{R}; L^2(\Omega))$.

Let $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, be a family of processes acting in a Banach space X with symbol space Σ; then for any $\sigma \in \Sigma$

$$U_\sigma(t,s) \circ U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \text{for any } t \geq s \geq \tau,\ \tau \in \mathbb{R}, \tag{23}$$

$$U_\sigma(\tau,\tau) = \mathrm{Id}\left(\text{identity}\right), \quad \text{for any } \tau \in \mathbb{R}. \tag{24}$$

The operators $\{H(s)\}_{s\geq 0}$ are the translation semigroup on Σ; a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is called to satisfy the translation identity if

$$U_\sigma(t+s, s+\tau) = U_{H(s)\sigma}(t,\tau), \quad \text{for any } \sigma \in \Sigma,\ t \geq \tau,\ \tau \in \mathbb{R},\ s \geq 0, \tag{25}$$

$$H(s)\Sigma = \Sigma, \quad \text{for any } s \geq 0. \tag{26}$$

We recall (e.g., see [28]) also that the kernel $\mathscr{K}$ of the process $U(t,\tau)$ consists of all bounded complete trajectories of the process; that is,

$$\mathscr{K} = \left\{u(\cdot) : \|u(t)\|_X \leq C_u,\ u(t) = U(t,\tau)u(\tau),\ \forall t \geq \tau,\ \tau \in \mathbb{R}\right\} \tag{27}$$

and $\mathscr{K}(s) = \{u(s) : u(\cdot) \in \mathscr{K}\} \subset X$ is denoted by the kernel section at a time moment $s \in \mathbb{R}$.

It is well-known that the key point is to obtain certain asymptotic compactness for the solution operator in the study of the long time behavior, especially for attractors. The nonlinearity having an arbitrary polynomial growth brings a difficulty here even for the autonomous and without memory case; see, for example, [22, 24, 27]. The main contribution of this paper is to extend the method in [15, 22] to overcome the difficulty caused by a lack of Sobolev compact embedding theorems. The conception, asymptotic contractive function, and new a priori estimates for verifying uniform asymptotic compactness of the family of processes are devised. We also prove some *weak* continuity for the family of processes and then obtain the structure of the compact uniform attractors.

The main results of this paper are given expression to in the following two theorems, which will be proved in Sections 2 and 3, respectively.

Theorem 2. *Let X be a Banach space and let $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, be a family of processes on X. Assume further that $g_0 \in L^2_b(\mathbb{R}; L^2(\Omega))$ and Σ is the hull of g_0 in $L^2_{w,loc}(\mathbb{R}; L^2(\Omega))$. Then $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, has a uniform attractor in X provided that the following conditions hold true:*

(i) *$\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, has a bounded absorbing set B_0 in X.*

(ii) *For any $\varepsilon > 0$, there exist $T = T(B_0, \varepsilon) > 0$ and an asymptotic contractive function ψ_T on $B_0 \times B_0$ such that*

$$\left\|U_{\sigma_1}(T,\tau)x - U_{\sigma_2}(T,\tau)y\right\|_X \leq \varepsilon + \psi_T\left(x, y, \sigma_1, \sigma_2\right), \quad \forall x, y \in B_0,\ \sigma_1, \sigma_2 \in \Sigma, \tag{28}$$

where ψ_T depend on T.

Theorem 3 (uniform attractor). *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and μ and f satisfy (4)–(6) and (13)-(14), respectively. Assume further that $g_0 \in L^2_b(\mathbb{R}; L^2(\Omega))$ and Σ is the hull of g_0 in $L^2_{w,loc}(\mathbb{R}; L^2(\Omega))$. Then the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, corresponding to (10)-(11), has a compact uniform (with respect to $\sigma \in \Sigma$) attractor $\mathscr{A}$ in $\mathscr{M}_1$. Moreover, this attractor can be decomposed as follows:*

$$\mathscr{A} = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(0), \tag{29}$$

where $\mathscr{K}_\sigma$ is the kernel of the process U_σ and $\mathscr{K}_\sigma(0)$ is the kernel section at time 0.

This theorem gives the existence of the uniform attractor and its structure as the union of the kernel sections (see [29]) of the nonautonomous process.

It is worth noting that Theorem 3 is also interesting in the autonomous case (see, e.g., [5]) or in the unbounded domain case (see, e.g., [22, 23]). This is the basis for further considering the asymptotic behavior, such as, to construct the nonautonomous (pullback) exponential attractor [2, 25]. On the other hand, from Theorem 16, the existence of the uniform attractor is obtained directly. However, since the external forcing term $g(x,t)$ is only assumed to be *translation-bounded*, consequently the symbol space is only weak compact with respect to the local *weak* convergence topology. So, in order to obtain the structure equality (29), we need to verify some *weak* continuity for the family of processes, which is different from the usual *strong* continuity. This may be the reason why some authors (e.g., see [1]) have to assume further that the external forcing term $g(x,t)$ is *translation-compact*.

2. Preliminaries

Definition 4 (see [15]). Let X be a Banach space, B be a bounded subset of X and Σ be a symbol (or parameter) space. We call a function $\phi(\cdot,\cdot,\cdot,\cdot)$, defined on $(X\times X)\times(\Sigma\times\Sigma)$, to be a contractive function on $B\times B$, if, for any sequence $\{x_n\}_{n=1}^{\infty}\subset B$, and any $\{\sigma_n\}_{n=1}^{\infty}\subset\Sigma$, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}\subset\{x_n\}_{n=1}^{\infty}$ and $\{\sigma_{n_k}\}_{k=1}^{\infty}\subset\{\sigma_n\}_{n=1}^{\infty}$ respectively, such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi\left(x_{n_k},x_{n_l},\sigma_{n_k},\sigma_{n_l}\right)=0. \tag{30}$$

Definition 5. Let X be a Banach space and let B be a bounded subset of X and let Σ be a symbol (or parameter) space. We call a function $\psi(\cdot,\cdot,\cdot,\cdot)$, defined on $(X\times X)\times(\Sigma\times\Sigma)$, to be an asymptotic contractive function on $B\times B$, if for any $\varepsilon>0$ and $x_i\in B,\sigma_i\in\Sigma$ $(i=1,2)$, there is a contractive function ϕ on $B\times B$ such that

$$\psi\left(x_1,x_2,\sigma_1,\sigma_2\right)\le\varepsilon+\phi\left(x_1,x_2,\sigma_1,\sigma_2\right). \tag{31}$$

We denote the set of all asymptotic contractive functions on $B\times B$ by $\mathfrak{C}(B,\Sigma)$.

In the following theorem, we present a new method (or technical) to verify the asymptotic compactness for the family of processes generated by evolutionary equations, which will be used in our later discussion.

Theorem 6. *Let $(X,\|\cdot\|_X)$ be a Banach space; $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, is a family of processes on X. Assume further that the following conditions hold true:*

(i) *$\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, has a bounded absorbing set B_0 in X.*

(ii) *For any $\varepsilon>0$ and $x,y\in B_0$, $\sigma_1,\sigma_2\in\Sigma$, there exist $T=T(B_0,\varepsilon)>0$ and $\psi_T\in\mathfrak{C}(B_0,\Sigma)$ such that*

$$\left\|U_{\sigma_1}(T,\tau)x-U_{\sigma_2}(T,\tau)y\right\|_X\le\varepsilon+\psi_T\left(x,y,\sigma_1,\sigma_2\right). \tag{32}$$

Then $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, is uniform (with respect to $\sigma\in\Sigma$) asymptotically compact in X, where ψ_T depend on T.

Proof. For any $\tau\in\mathbb{R}$, we assume that B is any bounded subset of X, and $\{x_n\}_{n=1}^{\infty}\subset B$, $\sigma_n\in\Sigma$ and $t_n\ge\tau$ satisfy $t_n\to+\infty$ as $n\to\infty$. It is enough to show that, from the assumptions, there is $N>0$ such that $U_\sigma(t_N,\tau)x\in B_0$ and $t_n>t_N-\tau$ for each $n>N$ large enough and $x\in B$ and furthermore that

$$\begin{aligned}U_\sigma\left(t_n,\tau\right)x&=U_\sigma\left(t_n,t_N\right)\circ U_\sigma\left(t_N,\tau\right)x\\&=U_\sigma\left(t_n,t_N\right)y=U_{\sigma'}\left(t_n',\tau\right)y,\end{aligned} \tag{33}$$

where $y=U_\sigma(t_N,\tau)x\in B_0$, $t_n'=t_n-t_N+\tau>0$ and $\sigma'=H(t_N-\tau)\sigma\in\Sigma$. So we only need to consider the case as $\{x_n\}_{n=1}^{\infty}\subset B_0$.

In the following, the existence of a Cauchy subsequence of $\{U_{\sigma_n}(t_n,\tau)x_n\}_{n=1}^{\infty}$ is proved by the diagonal method.

Take $\varepsilon_m>0$ with $\varepsilon_m\to0$ as $m\to\infty$.

At first, for ε_1, by the assumptions, there exist $T_1=T_1(\varepsilon_1)$ and $\psi_{T_1}\in\mathfrak{C}(B_0,\Sigma)$ such that

$$\begin{aligned}&\left\|U_{\sigma_1}\left(T_1,\tau\right)x-U_{\sigma_2}\left(T_1,\tau\right)y\right\|_X\\&\quad\le\varepsilon_1+\psi_{T_1}\left(x,y,\sigma_1,\sigma_2\right)\\&\qquad\text{for any } x,y\in B_0,\ \sigma_1,\sigma_2\in\Sigma.\end{aligned} \tag{34}$$

Since $t_n\to+\infty$, for some T_1 fixed we assume that $t_n\gg T_1$ is so large that $U_{g_n}(t_n-T_1+\tau,\tau)x_n\in B_0$ for each $n\ge N$ and $g_n\in\Sigma$.

Let $y_n=U_{g_n}(t_n-T_1+\tau,\tau)x_n$; then from (23)–(26) we have

$$\begin{aligned}&\left\|U_{g_n}\left(t_n,\tau\right)x_n-U_{g_m}\left(t_m,\tau\right)x_m\right\|_X\\&\quad=\left\|U_{g_n}\left(t_n,t_n-T_1+\tau\right)\circ U_{g_n}\left(t_n-T_1+\tau,\tau\right)x_n\right.\\&\quad-U_{g_m}\left(t_m,t_m-T_1+\tau\right)\\&\quad\left.\circ U_{g_m}\left(t_m-T_1+\tau,\tau\right)x_m\right\|_X=\left\|U_{\sigma_n}\left(T_1,\tau\right)y_n\right.\\&\quad\left.-U_{\sigma_m}\left(T_1,\tau\right)y_m\right\|_X\le\varepsilon_1+\psi_{T_1}\left(y_n,y_m,\sigma_n,\sigma_m\right),\end{aligned} \tag{35}$$

where $\sigma_n=H(t_n-T_1)g_n$.

Due to the definition of $\mathfrak{C}(B_0,\Sigma)$ and $\psi_{T_1}\in\mathfrak{C}(B_0,\Sigma)$, we know that, for ε_1 from (35), there is a contractive function ϕ on $B_0\times B_0$ and $\{(y_n,\sigma_n)\}_{n=1}^{\infty}$ has a subsequence $\{(y_{n_k}^{(1)},\sigma_{n_k}^{(1)})\}_{k=1}^{\infty}$ such that one gets the following estimation:

$$\begin{aligned}&\psi_{T_1}\left(y_{n_k}^{(1)},y_{n_l}^{(1)},\sigma_{n_k}^{(1)},\sigma_{n_l}^{(1)}\right)\\&\quad\le\varepsilon_1+\phi_{T_1}\left(y_{n_k}^{(1)},y_{n_l}^{(1)},\sigma_{n_k}^{(1)},\sigma_{n_l}^{(1)}\right),\end{aligned} \tag{36}$$

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi_{T_1}\left(y_{n_k}^{(1)},y_{n_l}^{(1)},\sigma_{n_k}^{(1)},\sigma_{n_l}^{(1)}\right)=0. \tag{37}$$

And similar to [15, 22, 30], we have

$$\begin{aligned}&\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\left\|U_{\sigma_{n_{k+p}}^{(1)}}\left(t_{n_{k+p}}^{(1)},\tau\right)x_{n_{k+p}}^{(1)}-U_{\sigma_{n_k}^{(1)}}\left(t_{n_k}^{(1)},\tau\right)x_{n_k}^{(1)}\right\|_X\\&\quad\le\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\limsup_{l\to\infty}\left\|U_{\sigma_{n_{k+p}}^{(1)}}\left(t_{n_{k+p}}^{(1)}\right)x_{n_{k+p}}^{(1)}\right.\\&\quad\left.-U_{\sigma_{n_l}^{(1)}}\left(t_{n_l}^{(1)},\tau\right)x_{n_l}^{(1)}\right\|_X\\&\quad+\limsup_{k\to\infty}\limsup_{l\to\infty}\left\|U_{\sigma_{n_k}^{(1)}}\left(t_{n_k}^{(1)},\tau\right)x_{n_k}^{(1)}\right.\\&\quad\left.-U_{\sigma_{n_l}^{(1)}}\left(t_{n_l}^{(1)},\tau\right)x_{n_l}^{(1)}\right\|_X\le\varepsilon_1\\&\quad+\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\lim_{l\to\infty}\psi_{T_1}\left(y_{n_{k+p}}^{(1)},y_{n_l}^{(1)},\sigma_{n_{k+p}}^{(1)},\sigma_{n_l}^{(1)}\right)+\varepsilon_1\\&\quad+\lim_{k\to\infty}\lim_{l\to\infty}\psi_{T_1}\left(y_{n_k}^{(1)},y_{n_l}^{(1)},\sigma_{n_k}^{(1)},\sigma_{n_l}^{(1)}\right)\le4\varepsilon_1\\&\quad+\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\lim_{l\to\infty}\phi_{T_1}\left(y_{n_{k+p}}^{(1)},y_{n_l}^{(1)},\sigma_{n_{k+p}}^{(1)},\sigma_{n_l}^{(1)}\right)\\&\quad+\lim_{k\to\infty}\lim_{l\to\infty}\phi_{T_1}\left(y_{n_k}^{(1)},y_{n_l}^{(1)},\sigma_{n_k}^{(1)},\sigma_{n_l}^{(1)}\right)\end{aligned} \tag{38}$$

which, combined with (35) and (37), implies that

$$\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\left\|U_{\sigma^{(1)}_{n_{k+p}}}\left(t^{(1)}_{n_{k+p}},\tau\right)x^{(1)}_{n_{k+p}}-U_{\sigma^{(1)}_{n_k}}\left(t^{(1)}_{n_k},\tau\right)x^{(1)}_{n_k}\right\|_X \le 5\varepsilon_1. \tag{39}$$

Therefore, there is K_1 such that

$$\left\|U_{\sigma^{(1)}_{n_k}}\left(t^{(1)}_{n_k},\tau\right)x^{(1)}_{n_k}-U_{\sigma^{(1)}_{n_l}}\left(t^{(1)}_{n_l},\tau\right)x^{(1)}_{n_l}\right\|_X \le 6\varepsilon_1 \quad \forall k,l\ge K_1. \tag{40}$$

By induction, we obtain that, for each $m\ge 1$, there is a subsequence $\{U_{\sigma^{(m+1)}_{n_k}}(t^{(m+1)}_{n_k},\tau)x^{(m+1)}_{n_k}\}_{k=1}^{\infty}$ of $\{U_{\sigma^{(m)}_{n_k}}(t^{(m)}_{n_k},\tau)x^{(m)}_{n_k}\}_{k=1}^{\infty}$ and certain K_{m+1} such that

$$\left\|U_{\sigma^{(m+1)}_{n_k}}\left(t^{(m+1)}_{n_k},\tau\right)x^{(m+1)}_{n_k}-U_{\sigma^{(m+1)}_{n_l}}\left(t^{(m+1)}_{n_l},\tau\right)x^{(m+1)}_{n_l}\right\|_X \le 6\varepsilon_{m+1} \tag{41}$$

holds for all $k,l\ge K_{m+1}$. Now, we consider the diagonal subsequence $\{U_{\sigma^{(k)}_{n_k}}(t^{(k)}_{n_k},\tau)x^{(k)}_{n_k}\}_{k=1}^{\infty}$. Since, for each $m\in\mathbb{N}$, $\{U_{\sigma^{(k)}_{n_k}}(t^{(k)}_{n_k},\tau)x^{(k)}_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{U_{\sigma^{(m)}_{n_k}}(t^{(m)}_{n_k},\tau)x^{(m)}_{n_k}\}_{k=1}^{\infty}$, then

$$\left\|U_{\sigma^{(k)}_{n_k}}\left(t^{(k)}_{n_k},\tau\right)x^{(k)}_{n_k}-U_{\sigma^{(l)}_{n_l}}\left(t^{(l)}_{n_l},\tau\right)x^{(l)}_{n_l}\right\|_X \le 6\varepsilon_m \quad \forall k,l\ge\max\{m,K_m\}, \tag{42}$$

which, combined with $\varepsilon_m\to 0$ as $m\to\infty$, implies that $\{U_{\sigma^{(k)}_{n_k}}(t^{(k)}_{n_k},\tau)x^{(k)}_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence in X. This shows that $\{U_{\sigma_n}(t_n,\tau)x_n\}_{n=1}^{\infty}$ is precompact in X. Then the proof is complete. □

Proof of Theorem 2. From Theorem 6, the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, on X satisfy the following conditions:

(i) $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, has a bounded absorbing set B_0 in X;

(ii) $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, is asymp

(iii) totically compact in X.

Then $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, has a uniform attractor in X. □

Lemma 7 (see [31]). *Let $X\Subset H\subset Y$ be Banach spaces, with X reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0,T;X)$ and du_n/dt is uniformly bounded in $L^p(0,T;Y)$, for some $p>1$. Then there is a subsequence of u_n that converges strongly in $L^2(0,T;H)$.*

3. Uniform Attractors in $H_0^1(\Omega)\times L^2_\mu(\mathbb{R}^+;H_0^1(\Omega))$

3.1. A Priori Estimates. We start with the following general existence and uniqueness of solutions for the nonclassical diffusion equations with fading memory which can be obtained by the *Galerkin* approximation methods; here we formulate only the results.

Lemma 8. *Assume that $g_0\in L^2_b(\mathbb{R};L^2(\Omega))$ and $g\in\Sigma$ and f satisfies (13)-(14). Then, for any initial data $z_\tau\in\mathscr{M}_1$ and any $T>0$, there exists a unique solution $z=(u,\eta^t)$ for problem (10)-(11). Moreover, we have the following Lipschitz continuity: for any (z^i_τ,g_i) $(z^i_\tau\in\mathscr{M}_1,g_i\in\Sigma)$, denote by z_i $(i=1,2)$ the corresponding solutions of (10); then for all $\tau\le t\le T+\tau$*

$$\|z_1(t)-z_2(t)\|^2_{\mathscr{M}_1}\le Q\left(\left\|z^1_\tau\right\|_{\mathscr{M}_1},\left\|z^2_\tau\right\|_{\mathscr{M}_1},T\right)\cdot\left(\left\|z^1_\tau-z^2_\tau\right\|^2_{\mathscr{M}_1}+\|g_1-g_2\|^2_{L^2_b}\right), \tag{43}$$

where $Q(\cdot)$ is a monotonically increasing function.

By Lemma 8, we can define a process $U_g(t,\tau)$ in $\mathscr{M}_1$ as the following:

$$U_g(t,\tau):\mathbb{R}_\tau\times\mathscr{M}_1\longrightarrow\mathscr{M}_1,\quad z_\tau=(u_\tau,\eta^\tau)\longrightarrow z(t)=(u(t),\eta^t)=U_g(t,\tau)z_\tau, \tag{44}$$

and $\{U_g(t,\tau)\}$, $g\in\Sigma$, is a family of processes on $\mathscr{M}_1$. See [5] for more details.

Lemma 9. *Let (13)-(14) hold, and $g\in\Sigma$. Then there exists a positive ρ_0, which depends only on $\|g\|_{L^2_b}$, such that, for any bounded subset $B\subset\mathscr{M}_1$, there is $T_0=T_0(\|B\|_{\mathscr{M}_1})$ such that*

$$\left\|U_g(t,\tau)z_\tau\right\|^2_{\mathscr{M}_1}\leqslant\rho_0,\quad \textit{for all } t-\tau\geqslant T_0 \textit{ and all } z_\tau\in B. \tag{45}$$

Proof. Multiplying (10) by u and then integrating over Ω, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(|u|_2^2+\|u\|_0^2+\left\|\eta^t\right\|^2_{\mu,\mathscr{V}}\right)+\|u\|_0^2+\left\langle\eta^t,\eta^t_s\right\rangle_{\mu,\mathscr{V}}+\langle f(u),u\rangle=\langle g,u\rangle. \tag{46}$$

Observe that

$$\langle f(u),u\rangle=\int_\Omega f(u)u\ge\gamma_1|u|_p^p-\beta_1\operatorname{mes}(\Omega). \tag{47}$$

Using Lemma 1, we have

$$\left\langle\eta^t,\eta^t_s\right\rangle_{\mu,\mathscr{V}}\ge\frac{\delta}{2}\left\|\eta^t\right\|^2_{\mu,\mathscr{V}}. \tag{48}$$

Using the Hölder inequality, combining with (47) and (48), then (46) can be reformulated as follows:

$$\frac{1}{2}\frac{d}{dt}\left(|u|_2^2+\|u\|_0^2+\left\|\eta^t\right\|^2_{\mu,\mathscr{V}}\right)+\frac{1}{2}\|u\|_0^2+\frac{\delta}{2}\left\|\eta^t\right\|^2_{\mu,\mathscr{V}}+\gamma_1|u|_p^p\le\beta_1\operatorname{mes}(\Omega)+\frac{1}{2\lambda_1^2}|g|_2^2. \tag{49}$$

According to Poincaré inequality, we get

$$\frac{1}{2}\frac{d}{dt}\left(|u|_2^2+\|u\|_0^2+\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2\right)+\frac{\lambda_1}{4}|u|_2^2+\frac{1}{4}\|u\|_0^2 + \frac{\delta}{2}\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2+\gamma_1|u|_p^p \le \frac{C}{2}\left(1+|g|_2^2\right). \tag{50}$$

Let $\alpha=\min(\lambda_1/2, 1/2, \delta)$; again, we have

$$\frac{d}{dt}\left(|u|_2^2+\|u\|_0^2+\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2\right) + \alpha\left(|u|_2^2+\|u\|_0^2+\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2\right)\le C\left(1+|g|_2^2\right). \tag{51}$$

Using the *Gronwall Lemma*, it follows from (51) that

$$\begin{aligned}|u|_2^2+\|u\|_0^2+\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2 &\le e^{-\alpha(t-\tau)}\left(|u_\tau|_2^2+\|u_\tau\|_0^2+\left\|\eta^\tau\right\|_{\mu,\mathscr{V}}^2\right)\\ &\quad + C\left(1+\|g\|_{L_b^2}^2\right).\end{aligned} \tag{52}$$

We know that $\|g\|_{L_b^2}^2$ is bounded, and by (12) and (6)

$$\begin{aligned}\left\|\eta^\tau\right\|_{\mu,\mathscr{V}}^2 &= \int_0^{+\infty}\mu(s)\left\|\eta^\tau(s)\right\|_0^2 ds\\ &\le \int_0^{+\infty}\int_0^s \mu(s)\left\|u_\tau(\tau-r)\right\|_0^2 dr\, ds\\ &\le \int_0^{+\infty} dr\int_r^{+\infty} e^{-\delta s}\left\|u_\tau(\tau-r)\right\|_0^2 ds\\ &\le \frac{1}{\delta}\int_0^{+\infty} e^{-\delta s}\left\|u_\tau(\tau-s)\right\|_0^2 ds \le \delta^{-1}\mathfrak{R}.\end{aligned} \tag{53}$$

So $\|z_\tau\|_{\mathscr{M}_1}^2=|u_\tau|_2^2+\|u_\tau\|_0^2+\|\eta^\tau\|_{\mu,\mathscr{V}}^2$ is bounded.

For any $t-\tau>T_0=(1/\alpha)\ln(\|z_\tau\|_{\mathscr{M}_1}^2/C(1+\|g\|_{L_b^2}^2))$, we infer from (52) that

$$|u(t)|_2^2+\|u(t)\|_0^2+\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2\le 2C\left(1+\|g\|_{L_b^2}^2\right). \tag{54}$$

For any $t-\tau>T_0$, we get

$$|u(t)|_2^2+\|u(t)\|_0^2+\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2\le \rho_0, \tag{55}$$

where $\rho_0=2C(1+\|g\|_{L_b^2}^2)$. Then we get

$$B_0=\left\{z=\left(u,\eta^t\right)\in\mathscr{M}_1:\|z\|_{\mathscr{M}_1}^2\le\rho_0\right\}. \tag{56}$$

The proof is complete. □

Combining with (43), we know that, for any $\tau\in\mathbb{R}$, U_g maps the bounded set of $\mathscr{M}_1$ into a bounded set for all $t\geqslant\tau$ that is as follows.

Corollary 10. *Let (13)-(14) hold, and $g\in\Sigma$. Then, for any bounded (in $\mathscr{M}_1$) subset B, there is $M_B=M(\|B\|_{\mathscr{M}_1},\|g\|_{L_b^2})$ such that*

$$\left\|U_g(t,\tau)z_\tau\right\|_{\mathscr{M}_1}^2\leqslant M_B \quad \text{for all } t\geqslant\tau \text{ and all } z_\tau\in B. \tag{57}$$

Lemma 11. *Let (13)-(14) hold, and $g\in\Sigma$. Then, for any bounded (in $\mathscr{M}_1$) subset B, there exists a positive constant $C=C(\alpha,\gamma_1,\rho_0)$, such that*

$$\begin{aligned}\int_t^{t+1}|u(s)|_p^p ds &< C\left(1+\|g\|_{L_b^2}^2\right),\\ \int_t^{t+1}\|z(s)\|_{\mathscr{M}_1}^2 ds &< C\left(1+\|g\|_{L_b^2}^2\right)\end{aligned} \tag{58}$$

hold for any $t-\tau\ge T_0$ (from Lemma 9).

Proof. Combining with (49), let $\alpha=\min(\lambda_1/2,1/2,\delta)$; it follows that

$$\begin{aligned}&\frac{d}{dt}\left(|u|_2^2+\|u\|_0^2+\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2\right)\\ &\quad+\alpha\left(|u|_2^2+\|u\|_0^2+\left\|\eta^t\right\|_{\mu,\mathscr{V}}^2\right)+\gamma_1|u|_p^p\\ &\le 2C\left(1+|g|_2^2\right).\end{aligned} \tag{59}$$

For any $t-\tau\ge T_0$ (from Lemma 9) and integrating the inequality above from t to $t+1$, we can get

$$\begin{aligned}&\alpha\int_t^{t+1}\|z(s)\|_{\mathscr{M}_1}^2 ds+2\gamma_1\int_t^{t+1}|u(s)|_p^p ds\\ &\quad\le\|z(t)\|_{\mathscr{M}_1}^2+2C\left(1+\|g\|_{L_b^2}^2\right).\end{aligned} \tag{60}$$

By Lemma 9, we have

$$\begin{aligned}&\alpha\int_t^{t+1}\|z(s)\|_{\mathscr{M}_1}^2 ds+2\gamma_1\int_t^{t+1}|u(s)|_p^p ds\\ &\quad\le\rho_0+2C\left(1+\|g\|_{L_b^2}^2\right).\end{aligned} \tag{61}$$

Setting $\alpha_0=\min(\alpha,2\gamma_1)$, then

$$\begin{aligned}\int_t^{t+1}\|z(s)\|_{\mathscr{M}_1}^2 ds &\le \frac{1}{\alpha_0}\left(\rho_0+2C\left(1+\|g\|_{L_b^2}^2\right)\right),\\ \int_t^{t+1}|u(s)|_p^p ds &\le \frac{1}{\alpha_0}\left(\rho_0+2C\left(1+\|g\|_{L_b^2}^2\right)\right).\end{aligned} \tag{62}$$

Let $C=C(\alpha,\gamma_1,\rho_0)=(1/\alpha_0)(\rho_0+2C)$; then we have that

$$\begin{aligned}\int_t^{t+1}\|z(s)\|_{\mathscr{M}_1}^2 ds &\le C\left(1+\|g\|_{L_b^2}^2\right),\\ \int_t^{t+1}|u(s)|_p^p ds &\le C\left(1+\|g\|_{L_b^2}^2\right)\end{aligned} \tag{63}$$

hold for any $t-\tau\ge T_0$. □

Let $F(s) = \int_0^s f(v)dv$; from assumption (3), there are positive constants $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\gamma}_1, \tilde{\gamma}_2$, such that

$$\tilde{\gamma}_1 |s|^p - \tilde{\beta}_1 \leq F(s) \leq \tilde{\gamma}_2 |s|^p + \tilde{\beta}_2. \quad (64)$$

Lemma 12. *There is a positive constant ρ_1, for any $z_\tau = (u_\tau, \eta^\tau) \in B_0$ (from Lemma 9); the following estimate*

$$|u(t)|_p^p \leq \rho_1 \quad (65)$$

holds for any $t - \tau \geq 1$.

Proof. Multiplying (10) by u_t and then integrating in Ω, we get

$$\begin{aligned} &|u_t(t)|_2^2 + \|u_t(t)\|_0^2 + \frac{d}{dt}\left(\frac{1}{2}\|u(t)\|_0^2 + \int_\Omega F(u(t))\right) \\ &\quad = \int_0^{+\infty} \mu(s)\left\langle \Delta\eta^t(s), u_t(t)\right\rangle ds + (g(t), u_t(t)). \end{aligned} \quad (66)$$

By the Hölder inequality and the Young inequality, we have

$$\begin{aligned} &\int_0^{+\infty} \mu(s)\left\langle \Delta\eta^t(s), u_t(t)\right\rangle ds \\ &\quad \leq \int_0^{+\infty} \mu(s)\left\|\eta^t(s)\right\|_0 \|u_t(t)\|_0 ds \\ &\quad \leq \frac{m_0}{2}\int_0^{+\infty} \mu(s)\left\|\eta^t(s)\right\|_0^2 ds \\ &\qquad + \frac{1}{2m_0}\|u_t(t)\|_0^2 \int_0^{+\infty} \mu(s) ds \\ &\quad \leq \frac{m_0}{2}\left\|\eta^t\right\|_{\mu,\mathcal{V}}^2 + \frac{1}{2}\|u_t(t)\|_0^2 \\ &\quad \leq \frac{m_0}{2}\|z(t)\|_{\mathcal{M}_1}^2 + \frac{1}{2}\|u_t(t)\|_0^2, \end{aligned} \quad (67)$$

here m_0 is from (7). And

$$(g(t), u_t(t)) \leq \frac{1}{2}|g(t)|_2^2 + \frac{1}{2}|u_t(t)|_2^2. \quad (68)$$

Let

$$E(t) = \frac{1}{2}\|u(t)\|_0^2 + \int_\Omega F(u(t)). \quad (69)$$

Then

$$\begin{aligned} &\frac{1}{2}|u_t(t)|_2^2 + \frac{1}{2}\|u_t(t)\|_0^2 + \frac{d}{dt}E(t) \\ &\quad \leq \frac{m_0}{2}\|z(t)\|_{\mathcal{M}_1}^2 + \frac{1}{2}|g(t)|_2^2. \end{aligned} \quad (70)$$

So we get the following inequality:

$$\frac{d}{dt}E(t) \leq \frac{m_0}{2}\|z(t)\|_{\mathcal{M}_1}^2 + \frac{1}{2}|g(t)|_2^2. \quad (71)$$

Integrating the inequality above to t from s $(t \leq s \leq t+1)$ to $t+1$, we have

$$E(t+1) \leq E(s) + \frac{m_0}{2}\int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds + \frac{1}{2}\|g\|_{L_b^2}^2, \quad (72)$$

and integrating (72) to s from t to $t+1$,

$$\begin{aligned} E(t+1) \leq{}& \int_t^{t+1} E(s) ds + \frac{m_0}{2}\int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds \\ &+ \frac{1}{2}\|g\|_{L_b^2}^2. \end{aligned} \quad (73)$$

Besides, by the Hölder inequality, Young inequality, and (13), we can infer from (69) that

$$\begin{aligned} E(s) &\leq \frac{1}{2}\|u(s)\|_0^2 + \tilde{\gamma}_2|u(s)|_p^p + \tilde{\beta}_2 \text{mes}(\Omega) \\ &\leq \frac{1}{2}\|z(s)\|_{\mathcal{M}_1}^2 + \tilde{\gamma}_2|u(s)|_p^p + \tilde{\beta}_2 \text{mes}(\Omega). \end{aligned} \quad (74)$$

On the contrary,

$$\begin{aligned} E(t+1) &\geq \frac{1}{2}\|u(t+1)\|_0^2 + \tilde{\gamma}_1|u(t+1)|_p^p \\ &\quad - \tilde{\beta}_1 \text{mes}(\Omega) \\ &\geq \tilde{\gamma}_1|u(t+1)|_p^p - \tilde{\beta}_1 \text{mes}(\Omega). \end{aligned} \quad (75)$$

Combining (73) and (74), we have

$$\begin{aligned} E(t+1) \leq{}& \frac{1+m_0}{2}\int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds \\ &+ \tilde{\gamma}_2\int_t^{t+1} |u(s)|_p^p ds + \frac{1}{2}\|g\|_{L_b^2}^2 \\ &+ \tilde{\beta}_2 \text{mes}(\Omega). \end{aligned} \quad (76)$$

By Lemma 11, it follows from (76) that

$$\begin{aligned} E(t+1) \leq{}& C\left(\frac{1+m_0}{2} + \tilde{\gamma}_2\right)\left(1 + \|g\|_{L_b^2}^2\right) + \frac{1}{2}\|g\|_{L_b^2}^2 \\ &+ \tilde{\beta}_2 \text{mes}(\Omega), \end{aligned} \quad (77)$$

and by (75)–(77), we have

$$\begin{aligned} |u(t+1)|_p^p \leq{}& \frac{1}{\tilde{\gamma}_1}\left(\left(\frac{1+m_0}{2} + \tilde{\gamma}_2\right)C\left(1 + \|g\|_{L_b^2}^2\right)\right. \\ &\left. + \frac{1}{2}\|g\|_{L_b^2}^2 + \left(\tilde{\beta}_1 + \tilde{\beta}_2\right)\text{mes}(\Omega)\right). \end{aligned} \quad (78)$$

Let

$$\begin{aligned} \rho_1 ={}& \frac{1}{\tilde{\gamma}_1}\left(\left(\frac{1+m_0}{2} + \tilde{\gamma}_2\right)C\left(1 + \|g\|_{L_b^2}^2\right) + \frac{1}{2}\|g\|_{L_b^2}^2\right. \\ &\left. + \left(\tilde{\beta}_1 + \tilde{\beta}_2\right)\text{mes}(\Omega)\right). \end{aligned} \quad (79)$$

Then the proof is complete. □

Lemma 13 (bounded uniformly absorbing set). *Let (13)-(14) hold, and $g \in \Sigma$. Then there exist positive ρ_0, ρ_1, which depend only on $\|g\|_{L_b^2}$, such that, for any bounded (in $\mathscr{M}_1$) subset B, there is $T_1 = T_1(\|B\|_{\mathscr{M}_1}) = T_0 + 1$ such that*

$$\begin{gathered}\left\|U_g(t,\tau) z_\tau\right\|_{\mathscr{M}_1}^2 \leqslant \rho_0, \\ |u(t)|_p^p \leq \rho_1\end{gathered} \tag{80}$$

hold for all $t-\tau \geqslant T_1$ and all $z_\tau \in B$.

For brevity, in the sequel, let B_0 be the bounded absorbing set obtained in Lemmas 9 and 12; that is,

$$B_0 = \left\{z = \left(u, \eta^t\right) \in \mathscr{M}_1 : \|z\|_{\mathscr{M}_1}^2 \leq \rho_0,\ |u|_p^p \leq \rho_1\right\}. \tag{81}$$

Lemma 14. *There is a positive constant ρ_2, for any $z_\tau = (u_\tau, \eta^\tau) \in B_0$ (from (81)); the following estimate*

$$\int_t^{t+1} \left(\left|u_t(s)\right|_2^2 + \left\|u_t(s)\right\|_0^2\right) ds \leq \rho_2, \tag{82}$$

holds for any $t-\tau \geq 0$.

Proof. By Lemma 12, for any $t-\tau \geq 0$, we integrate (70) to t from t to $t+1$; then we have

$$E(t+1) + \int_t^{t+1} \left(\left|u_t(s)\right|_2^2 + \left\|u_t(s)\right\|_0^2\right) ds \leq E(t). \tag{83}$$

According to Lemma 13, Corollary 10, and Lemma 12, we know that $E(t)$ is bounded for any $t-\tau \geq 0$, so there is a positive constant ρ_2, and the conclusion is true. □

3.2. Uniform Attractors. In the following, we will prove the existence of uniform attractors for system (10) with initial-boundary conditions (11) in $\mathscr{M}_1$ by using the method of asymptotic contractive function.

Lemma 15. *Let f satisfy (13)-(14); $z_n(t) = (u_n(t), \eta^{tn})$ is the sequence of solutions of (10) with initial data $z_{\tau n} = (u_{\tau n}, \eta^{\tau n}) \in B_0$ and $g_n \in \Sigma$ $(n = 1, 2, \ldots)$; then there is a subsequence u_{n_k} of $u_n(t)$ that converges strongly in $L^2(0,T;L^2(\Omega))$ and*

$$\begin{aligned}\lim_{k\to\infty}\lim_{j\to\infty} \int_0^T \int_\Omega &\left(g_{n_k}(x,s) - g_{n_j}(x,s)\right) \\ &\cdot \left(u_{n_k}(x,s) - u_{n_j}(x,s)\right) ds = 0.\end{aligned} \tag{84}$$

Proof. By Lemmas 13 and 14, the sequence u_n is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and du_n/dt is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$. Since $H_0^1(\Omega)$ is reflexive, is $L^2(0,T;H_0^1(\Omega))$. Let $X = H_0^1(\Omega)$, $H = L^2(\Omega)$, and $Y = H^{-1}(\Omega)$; then there is a subsequence $u_{n_k}(t)$ of $\{u_n(t)\}$ that converges strongly in $L^2(0,T;L^2(\Omega))$ by Lemma 7. One can write these as

$$\lim_{k\to\infty}\lim_{j\to\infty} \int_0^T \int_\Omega \left|u_{n_k}(x,s) - u_{n_j}(x,s)\right|^2 ds = 0. \tag{85}$$

Then by using Hölder inequality, we have

$$\begin{aligned}&\lim_{k\to\infty}\lim_{j\to\infty} \int_0^T \int_\Omega \left(g_{n_k}(x,s) - g_{n_j}(x,s)\right) \\ &\quad\cdot \left(u_{n_k}(x,s) - u_{n_j}(x,s)\right) ds \\ &\quad\leq \lim_{k\to\infty}\lim_{j\to\infty} \left(\int_0^T \int_\Omega \left|g_{n_k}(x,s) - g_{n_j}(x,s)\right|^2 ds\right)^{1/2} \\ &\quad\cdot \left(\int_0^T \int_\Omega \left|u_{n_k}(x,s) - u_{n_j}(x,s)\right|^2 ds\right)^{1/2} \leq 2T \|g\|_{L_b^2} \\ &\quad\cdot \lim_{k\to\infty}\lim_{j\to\infty} \left(\int_0^T \int_\Omega \left|u_{n_k}(x,s) - u_{n_j}(x,s)\right|^2 ds\right)^{1/2} \\ &\quad= 0.\end{aligned} \tag{86}$$

□

Theorem 16. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and let μ and f satisfy (4)-(5) and (13)-(14), respectively. Assume further that $g_0 \in L_b^2(\mathbb{R}; L^2(\Omega))$ and Σ is the hull of g_0 in $L_{w,loc}^2(\mathbb{R}; L^2(\Omega))$. $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is the family of processes generated by the solutions of (10) with initial data $z_\tau \in \mathscr{M}_1$. Then $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is uniformly (with respect to $\sigma \in \Sigma$) asymptotic compact in $\mathscr{M}_1$.*

Proof. For any $z_\tau^i = (u_\tau^i, \eta_i^\tau) \in B_0$ and $g^i \in \Sigma$, let $z_i(t) = (u^i(t), \eta_i^t)$ be the corresponding solution with respect to initial data z_τ^i $(i = 1, 2)$ and $g^i \in \Sigma$; that is, $z_i(t)$ is the solution of (10) with initial-boundary conditions (11).

For convenience, we denote $\omega(t) = u^1(t) - u^2(t)$ and $\xi^t = \eta_1^t - \eta_2^t$; then $\omega(t)$ satisfies the following equation:

$$\begin{aligned}&\omega_t(t) - \Delta\omega_t(t) - \Delta\omega(t) - \int_0^\infty \mu(s)\Delta\xi^t(s)\,ds \\ &\quad + f\left(u_1(t)\right) - f\left(u_2(t)\right) = g_1(t) - g_2(t) \\ &\xi_t^t = -\xi_s^t + \omega,\end{aligned} \tag{87}$$

with initial-boundary conditions

$$\begin{aligned}&\omega|_{\partial\Omega} = 0, \\ &\xi^t(x,s)\big|_{\partial\Omega\times\mathbb{R}^+} = 0, \quad t \geq \tau, \\ &\omega(x,\tau) = \omega_\tau(x) = u_\tau^1(x) - u_\tau^2(x), \quad x \in \Omega, \\ &\xi^\tau(x,s) = \eta_1^\tau(x,s) - \eta_1^\tau(x,s), \\ &\qquad (x,s) \in \Omega\times\mathbb{R}^+.\end{aligned} \tag{88}$$

Multiplying (87) by $\omega(t)$ and integrating in Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(|\omega|_2^2 + \|\omega\|_0^2 + \left\|\xi^t\right\|_{\mu,\mathscr{V}}^2\right) + \|\omega\|_0^2 \\ &\quad + \left\langle \xi^t(s), \xi_s^t(s)\right\rangle_{\mu,\mathscr{V}}\end{aligned}$$

$$+\int_{\Omega}\left(f\left(u_1\right)-f\left(u_2\right)\right)\left(u_1-u_2\right)$$
$$=\int_{\Omega}\left(g_1(t)-g_2(t)\right)\left(u_1-u_2\right). \tag{89}$$

By Lemma 1 and assumption (14), we have

$$\left\langle\xi^t(s),\xi_s^t(s)\right\rangle_{\mu,\mathscr{V}}\geq\frac{\delta}{2}\left\|\xi^t\right\|_{\mu,\mathscr{V}}^2,$$
$$\int_{\Omega}\left(f\left(u_1\right)-f\left(u_2\right)\right)\left(u_1-u_2\right)\geq-l|\omega|_2^2. \tag{90}$$

Using the Poincaré inequality, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(|\omega|_2^2+\|\omega\|_0^2+\left\|\xi^t\right\|_{\mu,\mathscr{V}}^2\right)+\frac{\lambda_1}{2}|\omega|_2^2+\frac{1}{2}\|\omega\|_0^2+\frac{\delta}{2}\left\|\xi^t\right\|_{\mu,\mathscr{V}}^2\leq l|\omega|_2^2+\int_{\Omega}\left(g_1(t)-g_2(t)\right)\left(u_1-u_2\right). \tag{91}$$

Let $\varrho=\min(\lambda_1,1,\delta)$; then we get

$$\frac{d}{dt}\left(|\omega|_2^2+\|\omega\|_0^2+\left\|\xi^t\right\|_{\mu,\mathscr{V}}^2\right)+\varrho\left(|\omega|_2^2+\|\omega\|_0^2+\left\|\xi^t\right\|_{\mu,\mathscr{V}}^2\right)\leq 2l\left|u_1-u_2\right|_2^2+2\int_{\Omega}\left(g_1(t)-g_2(t)\right)\left(u_1-u_2\right). \tag{92}$$

By the *Gronwall Lemma*, we have

$$|\omega(t)|_2^2+\|\omega(t)\|_0^2+\left\|\xi^t\right\|_{\mu,\mathscr{V}}^2\leq e^{-\varrho(t-\tau)}\left(\left|\omega_\tau\right|_2^2+\left\|\omega_\tau\right\|_0^2+\left\|\xi^\tau\right\|_{\mu,\mathscr{V}}^2\right)+2l\int_\tau^t e^{-\varrho(t-s)}\left|u_1(s)-u_2(s)\right|_2^2ds+2\int_\tau^t e^{-\varrho(t-s)}\int_{\Omega}\left(g_1(s)-g_2(s)\right)\cdot\left(u_1(s)-u_2(s)\right)ds. \tag{93}$$

For any $\varepsilon>0$, there exists $T_2=T_2(\varepsilon)=\max(T_1,(1/\varrho)\ln(\varepsilon/\|z_\tau\|_{\mathscr{M}_1}^2))\geq T_1$, such that

$$|\omega(t)|_2^2+\|\omega(t)\|_0^2+\left\|\xi^t\right\|_{\mu,\mathscr{V}}^2\leq\varepsilon+\psi_t \tag{94}$$

holds for any $t\geq T_2$; here

$$\psi_t=\psi_t\left(u_1,u_2,g_1,g_2\right)=2l\int_\tau^t e^{-\varrho(t-s)}\left|u_1(s)-u_2(s)\right|_2^2ds+2\int_\tau^t e^{-\varrho(t-s)}\int_{\Omega}\left(g_1(s)-g_2(s)\right)\cdot\left(u_1(s)-u_2(s)\right)ds. \tag{95}$$

Hereafter, we verify that there is $T\geq T_2$ such that ψ_T is an asymptotic contractive function on B_0. In fact

$$\begin{aligned}\psi_t&=2l\int_\tau^t e^{-\varrho(t-s)}\left|u_1(s)-u_2(s)\right|_2^2ds+2\int_\tau^t e^{-\varrho(t-s)}\int_{\Omega}\left(g_1(s)-g_2(s)\right)\cdot\left(u_1(s)-u_2(s)\right)ds\leq 2l\int_\tau^{T_1}e^{-\varrho(t-s)}\left|u_1(s)-u_2(s)\right|_2^2ds\\&+2\int_\tau^{T_1}e^{-\varrho(t-s)}\int_{\Omega}\left(g_1(s)-g_2(s)\right)\cdot\left(u_1(s)-u_2(s)\right)ds+2l\int_{T_1}^t e^{-\varrho(t-s)}\left|u_1(s)-u_2(s)\right|_2^2ds\\&+2\int_{T_1}^t e^{-\varrho(t-s)}\int_{\Omega}\left(g_1(s)-g_2(s)\right)\cdot\left(u_1(s)-u_2(s)\right)ds\\&\leq e^{-\varrho(t-T_1)}\left(2l\int_\tau^{T_1}\left|u_1(s)-u_2(s)\right|_2^2ds+2\int_\tau^{T_1}\int_{\Omega}\left(g_1(s)-g_2(s)\right)\left(u_1(s)-u_2(s)\right)ds\right)\\&+2l\int_{T_1}^t\left|u_1(s)-u_2(s)\right|_2^2ds+2\int_{T_1}^t\int_{\Omega}\left(g_1(s)-g_2(s)\right)\left(u_1(s)-u_2(s)\right)ds\\&\leq 2T_1e^{-\varrho(t-T_1)}\left((2l+1)M_B+\|g\|_{L_b^2}^2\right)+2l\int_0^{t-T_1}\left|u_1\left(s+T_1\right)-u_2\left(s+T_1\right)\right|_2^2ds\\&+2\int_0^{t-T_1}\int_{\Omega}\left(g_1\left(s+T_1\right)-g_2\left(s+T_1\right)\right)\cdot\left(u_1\left(s+T_1\right)-u_2\left(s+T_1\right)\right)ds,\end{aligned} \tag{96}$$

where M_B is from Corollary 10. Let

$$\varphi_t=2l\int_0^{t-T_1}\left|v_1(s)-v_2(s)\right|_2^2ds+2\int_0^{t-T_1}\int_{\Omega}\left(\sigma_1(s)-\sigma_2(s)\right)\left(v_1(s)-v_2(s)\right)ds, \tag{97}$$

where $v_i(s)=u_i(s+T_1)\in B_0$ and $\sigma_i(s)=g_i(s+T_1)\in\Sigma$ $(i=1,2)$. For any $\varepsilon>0$, let $t=T\geq T_1+(1/\varrho)\ln(2T_1((2l+1)M_B+\|g\|_{L_b^2}^2)/\varepsilon)$ be fixed; then

$$\psi_T\left(u_1,u_2,g_1,g_2\right)\leq\varepsilon+\varphi_T\left(u_1,u_2,g_1,g_2\right). \tag{98}$$

Combining Lemma 15 and Definition 5, we know that φ_T is the contractive function on $B_0 \times B_0$ and ψ_T is the asymptotic contractive function on $B_0 \times B_0$.

Then $\{U_g(t,\tau)\}_{t\geq 0}$, $g \in \Sigma$, is uniform (with respect to $g \in \Sigma$) asymptotic compact in $\mathscr{M}_1$ by Theorem 6. □

We also recall the following useful result, whose proof is simple and we omit it.

Lemma 17 (see [15]). *Let X be a reflexive Banach space and $x_n \rightharpoonup 0$ in X. Then, for each compact (in X^*) subset $B \subset X^*$, the uniform convergence holds: for any $\varepsilon > 0$, there is N_ε, depending only on ε, such that*

$$\left|\langle f, x_n \rangle_{X^*}\right| \leq \varepsilon \quad \text{for all } n \geq N_\varepsilon \text{ and all } f \in B. \tag{99}$$

Proof of Theorem 3. Theorem 2 implies that the family of processes $\{U_g(t,\tau)\}$, $g \in \Sigma$, corresponding to (1), has a compact (in $\mathscr{M}_1$) uniform (with respect to $g \in \Sigma$) attracting set which is bounded in $\mathscr{M}_1$; consequently, as a direct application of the abstract theorem [29, Chapter IV, Theorem 3.1], we obtain the existence of a compact uniform (with respect to $g \in \Sigma$) attractor $\mathscr{A}$, and $\mathscr{A} \subset \mathscr{M}_1$.

We remark that the above existence does not require any continuity of the family of processes (e.g., see [29]). However, in order to obtain the explicit form of $\mathscr{A}$, that is, equality (29), we need some continuity. Moreover, since the symbol space Σ now has only *weak compactness*, we need to verify the corresponding *weak continuity*.

Hence, in order to obtain (29), we need to verify the following: for any fixed $\tau \in \mathbb{R}$ and $t \geqslant \tau$, if $z_{n\tau} \to z_\tau$ in $\mathscr{M}_1$ and $g_n \rightharpoonup g$ with respect to the local weak convergence topology of $L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$, then

$$U_{g_n}(t,\tau) z_{n\tau} \longrightarrow U_g(t,\tau) z_\tau \quad \text{in } \mathscr{M}_1. \tag{100}$$

In the following, we will verify that the continuity above holds on $\mathscr{A} \times \Sigma$, which obviously is a compact uniformly attracting set for the corresponding skew-product flow semigroup (generated by the family processes $U_g(t,\tau)z_\tau$, $g \in \Sigma$, and the shift semigroup $\{T(t)\}_{t\geqslant 0}$). We will complete the proof of the week continuity.

Let $z_n(t) = (u_n, \eta_n^t)$, let $z_n(t)$ be the solutions of (10) with initial data $z_{\tau n} \in \mathscr{A}$ $(n = 1, 2, \ldots)$, and let the time-dependent forcing $g_n \in \Sigma$, $z(t)$ be the solutions of (10) with initial data $z_\tau \in \mathscr{A}$ and the time-dependent forcing $g \in \Sigma$; we assume the following hypotheses:

$$z_{\tau n} \longrightarrow z_\tau, \quad \text{in } \mathscr{M}_1, \tag{101}$$

$$g_n \longrightarrow g, \quad \text{in } L^2_{w,\mathrm{loc}}\left(\mathbb{R}, L^2(\Omega)\right). \tag{102}$$

Since (10) holds as an equality in $L^q(\tau, T; H^{-1}(\Omega))$ (q is conjugate to p from (13)), this means that for any $T(>\tau)$ fixed

$$u_n \rightharpoonup u, \quad \text{in } L^2\left(\tau, T; H_0^1(\Omega)\right), \tag{103}$$

where $u = \Pi_1 U_g(t,\tau) z_\tau \in L^2(\tau, T; H_0^1(\Omega))$, and Π_1 is the projector from $X \times Y$ to X. Since $\mathscr{A}$ is bounded in $\mathscr{M}_1$, following Corollary 10, we know that there is M such that

$$\sup_{g\in\Sigma} \sup_{\tau\in\mathbb{R}} \sup_{t\geqslant\tau} \left\|U_g(t,\tau)\mathscr{A}\right\|^2_{\mathscr{M}_1} \leqslant M < \infty. \tag{104}$$

Hence,

$$\begin{aligned} &\bigcup_{g\in\Sigma} \left\{\Pi_1 U_g(t,\tau) z_\tau : t \in [\tau, T],\ z_\tau \in \mathscr{A}\right\} \\ &\qquad \text{is bounded in } L^2\left(\tau, T; H_0^1(\Omega)\right), \\ &\bigcup_{g\in\Sigma} \left\{\partial_t \Pi_1 U_g(t,\tau) z_\tau : t \in [\tau, T],\ z_\tau \in \mathscr{A}\right\} \\ &\qquad \text{is bounded in } L^2\left(\tau, T; H^{-1}(\Omega)\right). \end{aligned} \tag{105}$$

Then by the compactness lemma (Lemma 7) we know that

$$\begin{aligned} &\bigcup_{g\in\Sigma} \left\{\Pi_1 U_g(t,\tau) z_\tau : t \in [\tau, T],\ z_\tau \in \mathscr{A}\right\} \\ &\qquad \text{is compact in } L^2\left(\tau, T; L^2(\Omega)\right). \end{aligned} \tag{106}$$

It is

$$\lim_{n\to\infty} \int_\tau^T \int_\Omega \left|u_n(s) - u(s)\right|^2 ds = 0. \tag{107}$$

Denote $\omega(t) = u_n(t) - u(t)$, $\xi^t = \eta_n^t - \eta^t$, $\omega_\tau = u_{n\tau}(t) - u_\tau$, and $z_n(t) = (u_n(t), \eta_n^t) = U_{g_n}(t,\tau) z_{n\tau} - U_g(t,\tau) z_\tau$, $z_{n\tau}, z_\tau \in \mathscr{A}$, $n = 1, 2, \ldots$. The proof of the following inequality is similar to (93):

$$\begin{aligned} &|\omega(t)|_2^2 + \|\omega(t)\|_0^2 + \left\|\xi^t\right\|^2_{\mu,\mathscr{V}} \\ &\quad \leq e^{-\varrho(t-\tau)} \left\|\omega_\tau\right\|^2_{\mathscr{M}_1} + 2l \int_\tau^t \left|u_n(s) - u(s)\right|_2^2 ds \\ &\quad + 2 \int_\tau^t \int_\Omega \left(g_n(s) - g(s)\right)\left(u_n(s) - u(s)\right) ds. \end{aligned} \tag{108}$$

Due to (107) and (102), we only need to show that if $g_n \to g$ in $L^2_{w,\mathrm{loc}}(\mathbb{R}; \mathbb{L}^2(\Omega))$, then

$$\left|\int_\tau^t \left\langle g_n(x,s) - g(x,s), w(x,s)\right\rangle ds\right| \longrightarrow 0 \tag{109}$$

uniformly on $L^2(\tau, t;\ L^2(\Omega))$-compact set. But this is a direct application of Lemma 17. Therefore we complete the proof of the *continuity* on $\mathscr{A} \times \Sigma$.

Based on the *continuity claim* above, and by constructing a skew-product flow on $\mathscr{A} \times \Sigma$ and applying [29, Chapter IV, Theorem 5.1], we obtain the structure equality (29). This completes the proof of Theorem 3. □

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Liouville Theorem for Some Elliptic Equations with Weights and Finite Morse Indices

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We establish the nonexistence of solution for the following nonlinear elliptic problem with weights: $-\Delta u = (1+|x|^{\alpha})|u|^{p-1}u$ in $\mathbb{R}^N$, where α is a positive parameter. Suppose that $1 < p < (N+2)/(N-2)$, $\alpha > (N-2)(p+1)/2 - N$ for $N \geq 3$ or $p > 1$, $\alpha > -2$ for $N = 2$; we will show that this equation does not possess nontrivial bounded solution with finite Morse index.

1. Introduction

Liouville theorems are very important in proving a priori bound of solutions for elliptic equations. As far as we know, the most powerful tool for proving the a priori bound is the blow-up method. During the blow-up process, we first suppose that the solutions are unbounded; then we can scale the sequence of solutions. Finally, we get a nontrivial solution for the limit equation. On the other hand, if we can prove that the limit equation does not possess a nontrivial solution, then we get a contradiction. So the solutions must be bounded. From the above statements, it is easy to see that the most important ingredient in proving the a priori bound is the nonexistence result for the limit equation. These kinds of nonexistence results are usually called Liouville type theorems.

For elliptic equations, the first Liouville theorem was proved by Gidas and Spruck in [1], in which the authors proved that the following equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N \tag{1}$$

does not possess positive solutions provided $0 < p < (N+2)/(N-2)$. Moreover, it was also proved that the exponent $(N+2)/(N-2)$ is optimal in the sense that problem (1) indeed possesses a positive solution for $p \geq (N+2)/(N-2)$. So the exponent $(N+2)/(N-2)$ is usually called the critical exponent for problem (1). However, this Liouville theorem is not sufficient for proving the a priori bound since the solutions may blow up on the boundary of the domain. In order to overcome this difficulty, the authors studied the limit equation in the half space

$$\begin{gathered} -\Delta u = u^p \quad \text{in } \mathbb{R}^N_+, \\ u = 0 \quad \text{on } \partial\mathbb{R}^N_+ \end{gathered} \tag{2}$$

in [2]. They proved the above equation also does not possess positive solutions provided $0 < p < (N+2)/(N-2)$. The above two Liouville theorems are what we need to prove a priori bound for positive solutions of nonlinear elliptic equations in bounded domain. Later, Chen and Li obtained similar nonexistence results for the above two equations in [3] by using the moving plane method.

At the same time, elliptic equations with weights

$$-\Delta u = |x|^{\alpha} u^p \quad \text{in } \mathbb{R}^N \tag{3}$$

were also widely studied and there are many existence and nonexistence results for problem (3). If $\alpha < 0$, we say this problem is a Hardy type equation, while for $\alpha > 0$, we say this problem is a Henon type equation. For the Hardy type problem, it can be proved that this problem does not possess positive solution provided $0 < p < (N + 2\alpha + 2)/(N - 2)$ by using the moving plane method as in [3]. However, for the Henon type equation, this proof of the nonexistence result is completely open up to now. The main difference between the two cases lies in that, for $\alpha < 0$, the weight $|x|^{\alpha}$ is decreasing in $|x|$, so the moving plane method works. However, for $\alpha > 0$, the weight $|x|^{\alpha}$ is increasing in $|x|$, so the moving plane method does not work.

On the other hand, we note that the above-mentioned results only claim that the above equations do not possess positive solution. A natural and more difficult question is whether the above equations possess sign-changing solution. However, this question is also completely open up to now. A partial answer was given in [4] in which the authors assume the solution has finite Morse index; then they proved the nonexistence result for this kind of solution. To prove this result, the author first deduced some integrable conditions on the solution based on finite Morse index; then they use the Pohozaev identity to prove the nonexistence result. After this work, there are many extensions on similar problems. For example, Harrabi et al. extended these results to more general nonlinear problems in [5, 6]. The corresponding Neumann boundary value problems were studied in [7]; Yu studied the mixed boundary problems, the nonlinear boundary value problem, and the fractional Laplacian equation in [8], [9], and [10], respectively.

In this paper, inspired by the above works, we study another problem, that is, the following elliptic equation with weight:

$$-\Delta u = \left(1 + |x|^{\alpha}\right)|u|^{p-1} u \quad \text{in } \mathbb{R}^N, \tag{4}$$

where $\alpha > 0$ is a positive parameter. We are mainly concerned with the nonexistence of solution with finite Morse index. Because of the interaction of $|u|^{p-1}u$ and $|x|^{\alpha}|u|^{p-1}u$, in order to prove the nonexistence result, we need to add a new bound for the exponent of the weight. More precisely, we have the following result.

Theorem 1. *Suppose that $1 < p < (N + 2)/(N - 2)$, $\alpha > (N-2)(p+1)/2 - N$ for $N \geq 3$ or $p > 1$, $\alpha > -2$ for $N = 2$; let u be a bounded solution for problem (4) with $i(u) < \infty$; then $u \equiv 0$, where $i(u)$ is the Morse index of u.*

The rest of this paper is devoted to the proof of the above theorem. We first deduce some inequality based on finite Morse index; then we derive some integral conditions on this solution. Finally, we use the Pohozaev inequality to prove the above theorem. In the following, we denote by C a positive constant, which may vary from line to line.

2. Proof of Theorem 1

In this section, we always assume the conditions in Theorem 1 hold. We establish the nonexistence of finite Morse index solution for problem (4). For this purpose, we first recall the definition of Morse index. Let u be a solution of problem (4); we define

$$\dim\left\{\varphi \in C_0^{\infty}\left(\mathbb{R}^N\right) \mid \left\langle I''(u)\varphi, \varphi\right\rangle < 0\right\} \tag{5}$$

as the Morse index for u, where

$$\begin{aligned} I(u) = {} & \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx \\ & - \frac{1}{p+1}\int_{\mathbb{R}^N}\left(1 + |x|^{\alpha}\right)|u|^{p+1}\, dx \end{aligned} \tag{6}$$

and hence

$$\begin{aligned} \left\langle I''(u)\varphi, \varphi\right\rangle = {} & \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx \\ & - p\int_{\mathbb{R}^N}\left(1 + |x|^{\alpha}\right)|u|^{p-1}\varphi^2 dx. \end{aligned} \tag{7}$$

Also, for further use, we need to define some cut-off function. Let $s > 2r > 0$; we define $0 \leq \phi_{r,s} \leq 1$ as follows:

$$\phi_{r,s} = \begin{cases} 0 & \text{for } |x| < r \text{ or } |x| > 2s, \\ 1 & \text{for } 2r \leq |x| \leq s. \end{cases} \tag{8}$$

Moreover, we assume that $|\nabla\phi_{r,s}| \leq 2/r$ for $r < |x| < 2r$ and $|\nabla\phi_{r,s}| \leq 2/s$ for $s < |x| < 2s$. In the same spirit of [8–10], we have the following result.

Lemma 2. *Let u be a solution of (4) with finite Morse index; then there exists $R_0 > 0$ such that*

$$\left\langle I''(u)\, u\phi_{R_0,R}, u\phi_{R_0,R}\right\rangle \geq 0 \tag{9}$$

for any $R > 2R_0$.

Proof. The proof is the same as [8–10]; we omit it. □

The next lemma is the key ingredient in the proof of Theorem 1.

Lemma 3. *Let u be a bounded solution of problem (4) with finite Morse index and let α, p satisfy the assumptions in Theorem 1; then one has*

$$\begin{gathered} \int_{\mathbb{R}^N} |u|^{p+1}\, dx < \infty, \\ \int_{\mathbb{R}^N} |x|^{\alpha}|u|^{p+1}\, dx < \infty, \\ \int_{\mathbb{R}^N} |\nabla u|^2\, dx < \infty. \end{gathered} \tag{10}$$

Proof. We will use the information of finite Morse index of u to prove our result. We first prove that $\int_{\mathbb{R}^N}|u|^{p+1}dx<\infty$. By Lemma 2, there exists a positive constant $R_0>0$, such that

$$\left\langle I''(u)\,u\phi_{R_0,R},u\phi_{R_0,R}\right\rangle\geq 0 \tag{11}$$

for any $R>2R_0$. Then by the definition of $I(u)$, we conclude that

$$\begin{aligned}p\int_{\mathbb{R}^N}&(1+|x|^{\alpha})|u|^{p+1}\phi_{R_0,R}^2dx\\&\leq\int_{\mathbb{R}^N}\left|\nabla\left(u\phi_{R_0,R}\right)\right|^2dx.\end{aligned}\tag{12}$$

A direct calculation shows that the right hand side of (12) equals

$$\int_{\mathbb{R}^N}|\nabla u|^2\phi_{R_0,R}^2+\left|\nabla\phi_{R_0,R}\right|u^2+2u\phi_{R_0,R}\nabla u\nabla\phi_{R_0,R}dx. \tag{13}$$

On the other hand, if we multiply (4) by $u\phi_{R_0,R}^2$ and integrate by parts, then we have

$$\begin{aligned}\int_{\mathbb{R}^N}&|\nabla u|^2\phi_{R_0,R}^2+2u\phi_{R_0,R}\nabla u\nabla\phi_{R_0,R}dx\\&=\int_{\mathbb{R}^N}(1+|x|^{\alpha})|u|^{p+1}\phi_{R_0,R}^2dx.\end{aligned}\tag{14}$$

Inserting (14) into (12), then we get

$$\begin{aligned}(p-1)\int_{\mathbb{R}^N}&(1+|x|^{\alpha})|u|^{p+1}\phi_{R_0,R}^2dx\\&\leq\int_{\mathbb{R}^N}u^2\left|\nabla\phi_{R_0,R}\right|^2dx\\&\leq C_0+\frac{C}{R^2}\int_{\{R\leq|x|\leq 2R\}\cap\mathbb{R}^N}u^2dx.\end{aligned}\tag{15}$$

In particular, we have

$$\begin{aligned}(p-1)\int_{\mathbb{R}^N}&|u|^{p+1}\phi_{R_0,R}^2dx\\&\leq C_0+\frac{C}{R^2}\int_{\{R\leq|x|\leq 2R\}\cap\mathbb{R}^N}u^2dx,\end{aligned}\tag{16}$$

$$\begin{aligned}(p-1)\int_{\mathbb{R}^N}&|x|^{\alpha}|u|^{p+1}\phi_{R_0,R}^2dx\\&\leq C_0+\frac{C}{R^2}\int_{\{R\leq|x|\leq 2R\}\cap\mathbb{R}^N}u^2dx.\end{aligned}\tag{17}$$

If $N=2$, we already have $\int_{\mathbb{R}^N}|u|^{p+1}\phi_{R_0,R}^2dx<\infty$ and $\int_{\mathbb{R}^N}|x|^{\alpha}|u|^{p+1}\phi_{R_0,R}^2dx<\infty$ since the right hand sides of (16) and (17) are bounded by a positive constant independent of R. This proves the result for $N=2$. So in the following, we always assume that $N\geq 3$. We deduce from (16) and the Holder inequality that

$$\begin{aligned}(p-1)\int_{\mathbb{R}^N}&|u|^{p+1}\phi_{R_0,R}^2dx\\&\leq C_0+\frac{C}{R^2}\int_{\{R\leq|x|\leq 2R\}\cap\mathbb{R}^N}u^2dx\\&\leq C_0+C\left(\int_{\{R\leq|x|\leq 2R\}\cap\mathbb{R}^N}|u|^{p+1}dx\right)^{2/(p+1)}\\&\quad\cdot R^{N((p-1)/(p+1))-2}.\end{aligned}\tag{18}$$

Suppose on the contrary that $\int_{\mathbb{R}^N}|u|^{p+1}dx$ is infinite; then we have

$$\begin{aligned}\int_{B_R}|u|^{p+1}dx&\leq C\left(\int_{B_{2R}}|u|^{p+1}dx\right)^{2/(p+1)}\\&\quad\cdot R^{N((p-1)/(p+1))-2}\end{aligned}\tag{19}$$

for some $C>0$ and R large enough. Denote $\mu=N((p-1)/(p+1))-2$, $\theta=2/(p+1)$, and $J(R)=\int_{B_R}|u|^{p+1}dx$; if we iterate the above inequality k times, then we get

$$J(R)\leq CR^{\mu\gamma}J\left(2^{k+1}R\right)^{\theta^{k+1}} \tag{20}$$

with $\gamma=1+\theta+\theta^2+\cdots+\theta^k$. Since u is bounded, a direct calculation shows that the right hand side of (20) is of order R^M with

$$M=\mu\frac{1-\theta^{k+1}}{1-\theta}+N\theta^{k+1}\longrightarrow\frac{\mu}{1-\theta} \tag{21}$$

as $k\to\infty$. In particular, we can choose k large enough, such that $M<0$. Then it follows from (20) that

$$J(R)\longrightarrow 0 \tag{22}$$

as $R\to\infty$, which is impossible. So we get $\int_{\mathbb{R}^N}|u|^{p+1}dx<\infty$.

Next, we show that $\int_{\mathbb{R}^N}|x|^{\alpha}|u|^{p+1}dx<\infty$. By the same spirit as the above, we deduce from (17) that

$$\begin{aligned}(p-1)\int_{\mathbb{R}^N}&|x|^{\alpha}|u|^{p+1}\phi_{R_0,R}^2dx\\&\leq C_0+\frac{C}{R^2}\int_{\{R\leq|x|\leq 2R\}\cap\mathbb{R}^N}u^2dx\\&\leq C_0+C\left(\int_{\{R\leq|x|\leq 2R\}\cap\mathbb{R}^N}|x|^{\alpha}|u|^{p+1}dx\right)^{2/(p+1)}\\&\quad\cdot R^{-2\alpha/(p+1)+N((p-1)/(p+1))-2}.\end{aligned}\tag{23}$$

Suppose on the contrary that $\int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} dx$ is infinite; then we have

$$\int_{B_R} |x|^\alpha |u|^{p+1} dx \le C \left(\int_{B_{2R}} |x|^\alpha |u|^{p+1} dx \right)^{2/(p+1)} \cdot R^{-2\alpha/(p+1)+N((p-1)/(p+1))-2} \tag{24}$$

for some $C > 0$ and R large enough. Denote $\widetilde{\mu} = -2\alpha/(p+1)+N((p-1)/(p+1))-2$ and $\widetilde{J}(R) = \int_{B_R} |x|^\alpha |u|^{p+1} dx$; similarly, if we iterate the above inequality k times, then we get

$$\widetilde{J}(R) \le C R^{\widetilde{\mu}\gamma} \widetilde{J}\left(2^{k+1}R\right)^{\theta^{k+1}} \tag{25}$$

with $\gamma = 1 + \theta + \theta^2 + \cdots + \theta^k$. By the boundedness of u, a direct calculation shows that the right hand side of (25) is of order $R^{\widetilde{M}}$ with

$$\widetilde{M} = \widetilde{\mu}\frac{1-\theta^{k+1}}{1-\theta} + (\alpha + N)\theta^{k+1} \longrightarrow \frac{\widetilde{\mu}}{1-\theta} \tag{26}$$

as $k \to \infty$. So we can still choose k large enough, such that $\widetilde{M} < 0$. Then it follows from (25) that

$$J(R) \longrightarrow 0 \tag{27}$$

as $R \to \infty$, which is a contradiction. So we get that $\int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} dx < \infty$.

Finally, we show that $\int_{\mathbb{R}^N} |\nabla u|^2 dx < \infty$. For this purpose, we first choose a cut-off function $0 \le \varphi \le 1$ such that $\varphi = 1$ for $|x| \le 1$ and $\varphi = 0$ for $|x| \ge 2$. For any $R > 0$, we multiply (4) by $u\varphi(x/R)$ and integrate by parts; then we get

$$\begin{aligned}\int_{B_{2R}} |u|^{p+1} \varphi\left(\frac{x}{R}\right) dx &= \int_{B_{2R}} -\Delta u u \varphi\left(\frac{x}{R}\right) dx \\ &= \int_{B_{2R}} |\nabla u|^2 \varphi\left(\frac{x}{R}\right) dx \\ &\quad + \frac{1}{R}\int_{B_{2R}} u \nabla u \nabla \varphi\left(\frac{x}{R}\right) dx \\ &= \int_{B_{2R}} |\nabla u|^2 \varphi\left(\frac{x}{R}\right) dx \\ &\quad - \frac{1}{2R^2}\int_{B_{2R}} u^2 \Delta\varphi\left(\frac{x}{R}\right) dx.\end{aligned} \tag{28}$$

Since

$$\begin{aligned}\frac{1}{2R^2}\int_{B_{2R}} u^2 \Delta\varphi\left(\frac{x}{R}\right) dx &\le \frac{C}{R^2}\int_{B_{2R}} u^2 dx \\ &\le C R^{-2+N((p-1)/(p+1))} \left(\int_{B_{2R}} |u|^{p+1} dx\right)^{2/(p+1)},\end{aligned} \tag{29}$$

we infer from the above two equations that

$$\begin{aligned}&\int_{B_{2R}} |\nabla u|^2 \varphi\left(\frac{x}{R}\right) dx \\ &\quad \le C R^{-2+N((p-1)/(p+1))} \left(\int_{B_{2R}} |u|^{p+1} dx\right)^{2/(p+1)} \\ &\quad + \int_{B_{2R}} |u|^{p+1} \varphi\left(\frac{x}{R}\right).\end{aligned} \tag{30}$$

By the assumption $p < (N+2)/(N-2)$ for $N \ge 3$ and the fact that $\int_{\mathbb{R}^N} |u|^{p+1} dx < \infty$, we get

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx < \infty \tag{31}$$

by letting $R \to \infty$ in the above equation. This finishes the proof of this lemma. □

In order to complete the proof of Theorem 1, we need the following Pohozaev identity for problem (4).

Lemma 4. *Suppose that u is a solution of (4); then the following identity holds:*

$$\begin{aligned}&\frac{N-2}{2}\int_{B_R} |\nabla u|^2 dx - \frac{N}{p+1}\int_{B_R} |u|^{p+1} dx \\ &\quad - \frac{N+\alpha}{p+1}\int_{B_R} |x|^\alpha |u|^{p+1} dx \\ &\quad = \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 dS - R\int_{\partial B_R} \left|\frac{\partial u}{\partial \nu}\right|^2 dS \\ &\quad - \frac{R}{p+1}\int_{\partial B_R} |u|^{p+1} dS \\ &\quad - \frac{R}{p+1}\int_{\partial B_R} |x|^\alpha |u|^{p+1} dS.\end{aligned} \tag{32}$$

Proof. The proof of this lemma is standard; we give the details to keep this paper self-contained.

Multiplying (4) by $\langle x, \nabla u\rangle$ and integrating in B_R, then we get

$$-\int_{B_R} \Delta u \langle x, \nabla u\rangle = \int_{B_R} \left(1 + |x|^\alpha\right) |u|^{p-1} u \langle x, \nabla u\rangle. \tag{33}$$

The left hand side of (33) equals

$$\begin{aligned}-\int_{B_R} \Delta u \langle x, \nabla u\rangle &= \int_{B_R} \nabla u \nabla \langle x, \nabla u\rangle dx \\ &\quad - \int_{\partial B_R} \frac{\partial u}{\partial \nu} \langle x, \nabla u\rangle dS \\ &= \int_{B_R} |\nabla u|^2 dx \\ &\quad + \frac{1}{2}\int_{B_R} \left\langle x, \nabla\left(|\nabla u|^2\right)\right\rangle dx\end{aligned}$$

$$
\begin{aligned}
&- \int_{\partial B_R} \frac{\partial u}{\partial \nu} \langle x, \nabla u \rangle \, dS \\
&= -\frac{N-2}{2} \int_{B_R} |\nabla u|^2 \, dx \\
&\quad + \frac{1}{2} \int_{\partial B_R} \langle x, \nu \rangle |\nabla u|^2 \, dS \\
&\quad - \int_{\partial B_R} \frac{\partial u}{\partial \nu} \langle x, \nabla u \rangle \, dS \\
&= -\frac{N-2}{2} \int_{B_R} |\nabla u|^2 \, dx \\
&\quad + \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 \, dS \\
&\quad - R \int_{\partial B_R} \left| \frac{\partial u}{\partial \nu} \right|^2 dS,
\end{aligned}
\tag{34}
$$

while the right hand side of (33) equals

$$
\begin{aligned}
&\int_{B_R} (1 + |x|^\alpha) |u|^{p-1} u \langle x, \nabla u \rangle \, dx \\
&= \frac{1}{p+1} \int_{B_R} \left\langle x, \nabla |u|^{p+1} \right\rangle dx \\
&\quad + \frac{1}{p+1} \int_{B_R} \left\langle |x|^\alpha x, \nabla |u|^{p+1} \right\rangle dx \\
&= -\frac{N}{p+1} \int_{B_R} |u|^{p+1} \, dx \\
&\quad + \frac{1}{p+1} \int_{\partial B_R} \langle x, \nu \rangle |u|^{p+1} \, dS \\
&\quad - \frac{N+\alpha}{p+1} \int_{B_R} |x|^\alpha |u|^{p+1} \, dx \\
&\quad + \frac{1}{p+1} \int_{\partial B_R} \langle x, \nu \rangle |x|^\alpha |u|^{p+1} \, dS \\
&= -\frac{N}{p+1} \int_{B_R} |u|^{p+1} \, dx + \frac{R}{p+1} \int_{\partial B_R} |u|^{p+1} \, dS \\
&\quad - \frac{N+\alpha}{p+1} \int_{B_R} |x|^\alpha |u|^{p+1} \, dx \\
&\quad + \frac{R}{p+1} \int_{\partial B_R} |x|^\alpha |u|^{p+1} \, dS.
\end{aligned}
\tag{35}
$$

Combining the above two equations together, then we get the above local Pohozaev identity for problem (4). □

With the above preparations, we can prove Theorem 1 now.

Proof of Theorem 1. First, since $\int_{\mathbb{R}^N} |u|^{p+1} dx < \infty$, $\int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} dx < \infty$, and $\int_{\mathbb{R}^N} |\nabla u|^2 dx < \infty$ by Lemma 3, then there exists a sequence $R_n \to \infty$ such that

$$
\begin{aligned}
\frac{R_n}{2} \int_{\partial B_{R_n}} |\nabla u|^2 \, dS &\longrightarrow 0, \\
R_n \int_{\partial B_{R_n}} \left| \frac{\partial u}{\partial \nu} \right|^2 dS &\longrightarrow 0, \\
\frac{R_n}{p+1} \int_{\partial B_{R_n}} |u|^{p+1} \, dS &\longrightarrow 0, \\
\frac{R_n}{p+1} \int_{\partial B_{R_n}} |x|^\alpha |u|^{p+1} \, dS &\longrightarrow 0
\end{aligned}
\tag{36}
$$

as $n \to \infty$. Let $R = R_n$ in the local Pohozaev identity and let $n \to \infty$; then we get

$$
\begin{aligned}
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx &= \frac{N}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx \\
&\quad + \frac{N+\alpha}{p+1} \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \, dx.
\end{aligned}
\tag{37}
$$

Next, multiplying (4) by u and integrating by parts, then we get

$$
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} |u|^{p+1} \, dx + \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \, dx. \tag{38}
$$

We infer from the above two identities that

$$
\begin{aligned}
&\left(\frac{N-2}{2} - \frac{N}{p+1} \right) \int_{\mathbb{R}^N} |u|^{p+1} \, dx \\
&= \left(\frac{N+\alpha}{p+1} - \frac{N-2}{2} \right) \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \, dx.
\end{aligned}
\tag{39}
$$

By the assumptions on the exponents p and α in Theorem 1, we have

$$
\begin{aligned}
\frac{N-2}{2} - \frac{N}{p+1} &< 0, \\
\frac{N+\alpha}{p+1} - \frac{N-2}{2} &> 0.
\end{aligned}
\tag{40}
$$

For (39) to hold, the only possibility is

$$
\int_{\mathbb{R}^N} |u|^{p+1} \, dx = 0, \tag{41}
$$

which finally implies $u \equiv 0$. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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Research Article

Trace Operators on Wiener Amalgam Spaces

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The paper deals with trace operators of Wiener amalgam spaces using frequency uniform decomposition operators and maximal inequalities, obtaining sharp results. Additionally, we provide the embedding between standard and anisotropic Wiener amalgam spaces.

1. Introduction

The aim of this paper is to study the trace problem: what can be said about the trace operator $\mathbb{T}$,

$$\mathbb{T} : f(x) \longrightarrow f(\bar{x}, 0), \quad \bar{x} = (x_1, x_2, \dots, x_{n-1}), \tag{1}$$

as a mapping from $W_s^{p,q}(\mathbb{R}^n)$ to $W_s^{p,q}(\mathbb{R}^{n-1})$. We note that, for a tempered distribution f defined on $\mathbb{R}^n$, $f(x, 0)$ has no straightforward meaning and the question is how to define the trace for a class of tempered distributions. One can resort to the Schwartz function ϕ, which has a pointwise trace $\phi(\bar{x}, 0)$. It can be extended to (quasi-)Banach function spaces which contain the Schwartz space $\mathcal{S}$ as a dense subspace.

Our setting is on Wiener amalgam spaces. These spaces, together with modulation spaces, were introduced by Feichtinger [1–3] in the 80s and are now widely used function spaces for various problems in PDE and harmonic analysis [4–10]. They resemble Triebel-Lizorkin spaces in the sense that we are taking $L^p(\ell^q)$ norms but differ with the decomposition operator being used. Instead of the dyadic decomposition operators $\Delta_k \sim \mathcal{F}^{-1}\chi_{\{\xi:|\xi|\sim 2^k\}}\mathcal{F}$ used for Triebel-Lizorkin spaces, Wiener amalgam spaces use frequency uniform decomposition operators $\square_k \sim \mathcal{F}^{-1}\chi_{Q_k}\mathcal{F}$, where Q_k denotes a unit cube with center k and $\cup_{k\in\mathbb{Z}^n} Q_k = \mathbb{R}^n$.

The concept of trace operator plays an important role in studying the existence and uniqueness of solutions to boundary value problems, that is, to partial differential equations with prescribed boundary conditions [11, 12]. The trace operator makes it possible to extend the notion of restriction of a function to the boundary of its domain to "generalized" functions in various function spaces with regularity. Now, we give a formal definition for the trace operators.

Definition 1. Let X and Y be quasi-Banach function spaces defined on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. Assume that the Schwartz class $\mathcal{S}$ is dense in X. Denote

$$\mathbb{T} : f(x) \longrightarrow f(\bar{x}, 0), \quad f \in \mathcal{S}. \tag{2}$$

Assuming that there exists a constant $C > 0$ such that

$$\|\mathbb{T}f\|_Y \le C\|f\|_X, \quad \forall f \in \mathcal{S}, \tag{3}$$

one can extend $\mathbb{T} : X \to Y$ by the density of $\mathcal{S}$ in X and we write $f(\bar{x}, 0) = \mathbb{T}f$, which is said to be the trace of $f \in X$. Moreover, if there exists a continuous linear operator $\mathbb{T}^{-1} : Y \to X$ such that $\mathbb{T}\mathbb{T}^{-1}$ is the identity operator on Y, then $\mathbb{T}$ is said to be a trace-retraction from X onto Y.

For (α-) modulation spaces, Besov spaces, and Triebel-Lizorkin spaces, trace theorems have been extensively studied [12–14]. Feichtinger et al. [13] considered the trace theorems on anisotropic modulation spaces $M_s^{p,q,r}$ with $0 < p, q, r < \infty$, $s \in \mathbb{R}$ and they obtained $\mathbb{T}M_s^{p,q,p\wedge q\wedge 1}(\mathbb{R}^n) = M_s^{p,q}(\mathbb{R}^{n-1})$. In [15, 16], we find that, for $0 < p, q \le \infty$, and $s - 1/p > (n-1)(1/p - 1)$, we have $\mathbb{T}B_s^{p,q}(\mathbb{R}^n) = B_{s-1/p}^{p,q}(\mathbb{R}^{n-1})$ and $\mathbb{T}F_s^{p,q}(\mathbb{R}^n) = F_{s-1/p}^{p,p}(\mathbb{R}^{n-1})$ (the case $F^{\infty,q}$ is omitted). The use of atoms as a framework in studying trace problems can be found in [16] and the references within.

Our main results are the following.

Theorem 2. *Let $n \geq 2$, $0 < p, q < \infty$, $s \in \mathbb{R}$. Then*

$$\mathbb{T} : f(x) \longrightarrow f(\overline{x}, 0), \quad \overline{x} = (x_1, x_2, \ldots, x_{n-1}) \tag{4}$$

is a trace-retraction from $W_s^{p,q,1\wedge q}(\mathbb{R}^n)$ to $W_s^{p,q}(\mathbb{R}^{n-1})$.

In view of the embedding in Theorem 6(II-ii), we immediately have the following corollary.

Corollary 3. *Let $n \geq 2$, $0 < p, q < \infty$, $s \geq 0$. Then for any $\epsilon > 0$*

$$\mathbb{T} : W^{p,q}_{s+1/(1\wedge q)-1/q+\epsilon}(\mathbb{R}^n) \longrightarrow W_s^{p,q}(\mathbb{R}^{n-1}). \tag{5}$$

We remark that Corollary 3 is an improvement of an older trace theorem found in [14] and that our result is sharp at least for $1 < p, q < \infty$. Moreover, our result shows independence of p. This is due to the pointwise estimates we were able to prove in Section 3. An interesting observation is that the trace theorem of Triebel-Lizorkin spaces stated above shows independence in q. This difference might be due to the decomposition operators used in the norm of each of the function spaces.

The paper is organised as follows. In Section 2, the embedding between standard and anisotropic Wiener amalgam spaces is given. We also define notations, function spaces, and some lemmas to be used throughout this paper. In Section 3, we prove our main result, Theorem 2, and the sharpness of Corollary 3.

2. Preliminaries

Notations. The Schwartz class of test functions on $\mathbb{R}^n$ will be denoted by $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ and its dual and the space of tempered distributions will be denoted by $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$. $L^p(\mathbb{R}^n)$ norm is given by $\|f\|_{L^p} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$ whenever $1 \leq p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|$. The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi x\cdot\xi} f(x)\, dx \tag{6}$$

which is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ onto itself that extends to the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ by duality. The inverse Fourier transform is given by $\mathcal{F}^{-1} f(x) = \check{f}(x) = \int_{\mathbb{R}^n} e^{i2\pi\xi\cdot x} f(\xi) d\xi$. Given $1 \leq p \leq \infty$, we denote by p' the conjugate exponent of p (i.e., $1/p + 1/p' = 1$). We use the notation $u \lesssim v$ to denote $u \leq cv$ for a positive constant c independent of u and v. We write $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. We now define the function spaces in this paper.

Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth bump function satisfying

$$\eta(\xi) := \begin{cases} 1, & |\xi| \leq 1 \\ \text{smooth}, & 1 < |\xi| \leq 2 \\ 0, & |\xi| \geq 2. \end{cases} \tag{7}$$

We write, for $k = (k_1, \ldots, k_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$,

$$\phi_{k_i} = \eta(2(\xi_i - k_i)). \tag{8}$$

Put

$$\varphi_k(\xi) = \frac{\phi_{k_1}(\xi_1)\cdots\phi_{k_n}(\xi_n)}{\sum_{k\in\mathbb{Z}^n} \phi_{k_1}(\xi_1)\cdots\phi_{k_n}(\xi_n)}, \quad k \in \mathbb{Z}^n. \tag{9}$$

Definition 4 (Wiener amalgam spaces). For $0 < p, q \leq \infty$, and $s \in \mathbb{R}$, the Wiener amalgam space $W_s^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'$ for which the following is finite:

$$\|f\|_{W_s^{p,q}} = \left\| \|\{\langle k\rangle^s \Box_k f\}\|_{\ell^q} \right\|_{L^p}, \tag{10}$$

with $\Box_k f = \mathcal{F}^{-1}(\varphi_k \widehat{f})$.

We note that (10) is a quasi-norm if $0 < p, q \leq \infty$ and norm if $1 \leq p, q \leq \infty$. Moreover, (10) is independent of the choice of $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}^n}$. We refer the reader to [1, 2, 17] for equivalent definitions (continuous versions).

We write $\overline{x} = (x_1, x_2, \ldots, x_{n-1})$ and define the anisotropic Wiener amalgam spaces $W^{p,q,r}$ by the following norm:

$$\|f\|_{W_s^{p,q,r}(\mathbb{R}^n)} = \left\| \left(\sum_{k_n\in\mathbb{Z}} \left(\sum_{\overline{k}\in\mathbb{Z}^{n-1}} \langle\overline{k}\rangle^{sq} |\Box_k f|^q \right)^{r/q} \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)}. \tag{11}$$

Similarly, for $\overline{\overline{x}} = (x_1, x_2, \ldots, x_{n-2})$, we define

$$\|f\|_{W_s^{p,q,r,r}(\mathbb{R}^n)} = \left\| \left(\sum_{(k_{n-1}, k_n)\in\mathbb{Z}^2} \left(\sum_{\overline{\overline{k}}\in\mathbb{Z}^{n-2}} \langle\overline{\overline{k}}\rangle^{sq} |\Box_k f|^q \right)^{r/q} \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)}. \tag{12}$$

Comparing amalgam spaces $W_s^{p,q}$ with anisotropic amalgam spaces $W_s^{p,q,r}$ we see that $W_s^{p,q}$ is rotational invariant but $W_s^{p,q,r}$ is not. Using the almost orthogonality of φ we see that $W_s^{p,q,r}$ is independent of φ. Moreover, recalling that $\|f\|_{W_s^{p,q,r}}$ is the function sequence $\{\Box_k f\}_{k\in\mathbb{Z}^n}$ equipped with the $L^p \ell^r_{k_n} \ell^q_{\overline{k}}$ norm, it is easy to see that $W_s^{p,q,r}$ is a quasi-Banach space for any $s \in \mathbb{R}$, $p, q, r \in (0, \infty]$ and a Banach space for any $s \in \mathbb{R}$, $1 \leq p, q, r \leq \infty$. Moreover, the Schwartz space is dense in $W_s^{p,q,r}$ if $p, q, r < \infty$. The proofs are similar to those of amalgam spaces in [1, 2, 17].

We collect properties of Wiener amalgam spaces in the following lemma.

Lemma 5. *Let $p, q, p_i, q_i \in [1, \infty]$ for $i = 1, 2$ and $s_j \in \mathbb{R}$ for $j = 1, 2$. Then one has the following:*

(1) *$\mathcal{S}(\mathbb{R}^n) \hookrightarrow W^{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$;*

(2) *$\mathcal{S}$ is dense in $W^{p,q}$ if p and $q < \infty$;*

(3) *if $q_1 \leq q_2$ and $p_1 \leq p_2$, then $W^{p_1,q_1} \hookrightarrow W^{p_2,q_2}$;*

(4) *if* $s_1 \geq s_2$, *then* $W^{p,q}_{s_1} \hookrightarrow W^{p,q}_{s_2}$;

(5) *(complex interpolation) let* $0 < \theta < 1$, $1/p = \theta/p_1 + (1-\theta)/p_2$, $1/q = \theta/q_1 + (1-\theta)/q_2$, *and* $s = \theta s_1 + (1-\theta)s_2$. *Then*

$$\left[W^{p_1,q_1}_{s_1}, W^{p_2,q_2}_{s_2}\right]_{[\theta]} = W^{p,q}_s. \tag{13}$$

The proofs of these statements can be found in [1, 3, 17, 18].

Theorem 6 (embedding: $W^{p,q}_s \hookrightarrow W^{p,q,r}_{s'}$). *Let* $p, q, r \in (0, \infty]$ *and* $s \geq 0$.

(I) *The case* $r = q$.

(I-i) *The case* $r = q = \infty$:

$$W^{p,\infty}_s \hookrightarrow W^{p,\infty,\infty}_s. \tag{14}$$

(I-ii) *The case* $r = q < \infty$:

$$W^{p,q}_s \hookrightarrow W^{p,q,q}_s. \tag{15}$$

(II) *The case* $r < q$.

(II-i) *The case* $q = \infty$. *If* $s > 1/r$, *then*

$$W^{p,\infty}_s \hookrightarrow W^{p,\infty,r}_{s'}, \tag{16}$$

for any $s' \in (-\infty, s - 1/r)$.

(II-ii) *The case* $q < \infty$. *If* $s > (1/r - 1/q)$, *then*

$$W^{p,q}_s \hookrightarrow W^{p,q,r}_{s'}, \tag{17}$$

for any $s' \in (-\infty, s - (1/r - 1/q))$.

(III) *The case* $q < r$.

(III-i) *The case* $r = \infty$:

$$W^{p,q}_s \hookrightarrow W^{p,q,\infty}_s. \tag{18}$$

(III-ii) *The case* $r < \infty$:

$$W^{p,q}_s \hookrightarrow W^{p,q,r}_s. \tag{19}$$

Proof. For part (I), it suffices to show the following estimates.

(I-i) Consider

$$\sup_{k_n} \sup_{\bar{k}} \langle \bar{k} \rangle^s |\Box_k f| \leq \sup_k \langle k \rangle^s |\Box_k f|. \tag{20}$$

(I-ii) Consider

$$\left(\sum_{k_n}\sum_{\bar{k}} \langle \bar{k} \rangle^{sq} |\Box_k f|^q\right)^{1/q} \leq \left(\sum_k \langle k \rangle^{sq} |\Box_k f|^q\right)^{1/q}. \tag{21}$$

(II-i) Let $s' := s - 1/r - \varepsilon$, $(\varepsilon > 0)$. We may assume that $s' \geq 0$:

$$\begin{aligned}&\left(\sum_{k_n} \sup_{\bar{k}} \langle \bar{k} \rangle^{s'r} |\Box_k f|^r\right)^{1/r} \\ &\leq \left(\sup_k \langle k \rangle^s |\Box_k f|\right)\left(\sum_{k_n} \sup_{\bar{k}} \langle k \rangle^{-sr} \langle \bar{k} \rangle^{s'r}\right)^{1/r}.\end{aligned} \tag{22}$$

The last term is equivalent to

$$\begin{aligned}&\left(\sum_{m\in\mathbb{Z}} \left(\sup_{t\geq 1} \left(\frac{1}{t + |m|}\right)^s t^{s'}\right)^r\right)^{1/r} \\ &\leq \left(\sum_{m\in\mathbb{Z}} \left(\frac{1}{1+|m|}\right)^{(s-s')r} \sup_{t\geq 1} t^{(s'-s')r}\right)^{1/r} < \infty,\end{aligned} \tag{23}$$

where $(s - s')r = 1 + \varepsilon r > 1$ and $s' \geq 0$ have been used.

(II-ii) Let $s' := s - (1/r - 1/q) - \varepsilon$, $(\varepsilon > 0)$. It suffices to show the embedding in the case $s' \geq 0$. Remark that $q/r \in (1, \infty)$ and $(q/r)' = 1/(r(1/r - 1/q))$. Let $\alpha := 1 - r/q + \varepsilon r$:

$$\begin{aligned}&\left[\sum_{k_n}\left\{\sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} |\Box_k f|^q\right\}^{r/q}\right]^{1/r} \\ &= \left[\sum_{k_n}\left\{\sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\Box_k f|^q\right\}^{r/q} \langle k_n \rangle^{-\alpha}\right]^{1/r} \\ &\leq \left[\left\{\sum_{k_n}\sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\Box_k f|^q\right\}^{r/q} \right. \\ &\quad \left. \cdot \left(\sum_{k_n} \langle k_n \rangle^{-\alpha(q/r)'}\right)^{1/(q/r)'}\right]^{1/r} \\ &\lesssim \left(\sum_k \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\Box_k f|^q\right)^{1/q} \\ &= \left\{\sum_k \langle k \rangle^{sq} |\Box_k f|^q \left(\langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s}\right)^q\right\}^{1/q} \\ &\leq \left[\sup_k \langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s}\right] \left\{\sum_k \langle k \rangle^{sq} |\Box_k f|^q\right\}^{1/q}.\end{aligned} \tag{24}$$

Here, we have used $\alpha(q/r)' = 1 + \varepsilon/(1/r - 1/q) > 1$. Because $\alpha/r = 1/r - 1/q + \varepsilon = s - s'$, $s - s' \geq 0$, and $s' \geq 0$,

$$\langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s} = \left(\frac{\langle k_n \rangle}{\langle k \rangle}\right)^{s-s'} \left(\frac{\langle \bar{k} \rangle}{\langle k \rangle}\right)^{s'} \lesssim 1. \tag{25}$$

(III-i) Consider

$$\sup_{k_n}\left(\sum_{\bar{k}}\langle\bar{k}\rangle^{sq}|\Box_k f|^q\right)^{1/q}\leq\left(\sum_{k}\langle k\rangle^{sq}|\Box_k f|^q\right)^{1/q}. \quad (26)$$

Here, we have used $s\geq 0$.

(III-ii) Using the embedding $\ell^q\hookrightarrow\ell^r$,

$$\begin{aligned}&\left\{\sum_{k_n}\left(\sum_{\bar{k}}\langle\bar{k}\rangle^{sq}|\Box_k f|^q\right)^{r/q}\right\}^{1/r}\\&\leq\left(\sum_{k}\langle\bar{k}\rangle^{sq}|\Box_k f|^q\right)^{1/q}\leq\left(\sum_{k}\langle k\rangle^{sq}|\Box_k f|^q\right)^{1/q}.\end{aligned} \quad (27)$$

In the last inequality, we need $s\geq 0$. □

Lemma 7 (Triebel [12]). *Let $0<p<\infty$ and $0<q\leq\infty$. Let $\Omega=\{\Omega_k\}_{k\in\mathbb{Z}^n}$ be a sequence of compact subsets of $\mathbb{R}^n$. Let d_k be the diameter of Ω_k. If $0<r<\min(p,q)$, then there exists a constant c such that*

$$\left\|\sup_{z\in\mathbb{R}^n}\frac{|f_k(\cdot-z)|}{1+|d_k z|^{n/r}}\right\|_{L^p(\ell^q)}\leq c\|f_k\|_{L^p(\ell^q)} \quad (28)$$

holds for all $f\in L^p_\Omega(\ell^q)$, where $f=\{f_k\}$, $\|f_k\|_{L^p(\ell^q)}=\|\|f_k(\cdot)\|_{\ell^q}\|_{L^p}$, and

$$\begin{aligned}L^p_\Omega(\ell^q)=\Big\{&f\mid f=\{f_k\}_{k\in\mathbb{Z}^n}\subset\mathcal{S}',\ \operatorname{supp}\mathcal{F}f_k\\&\subset\Omega_k,\ \|f_k\|_{L^p(\ell^q)}<\infty\Big\}.\end{aligned} \quad (29)$$

Definition 8 (maximal functions). Let $b>0$ and $f\in\mathcal{S}$. Then

$$\Box_k^* f(x):=\sup_{y\in\mathbb{Z}^n}\frac{|\Box_k f(x-y)|}{1+|y|^b}\quad x\in\mathbb{R}^n,\ k\in\mathbb{Z}^n. \quad (30)$$

Proposition 9. *Let $0<p<\infty$ and $0<q\leq\infty$, $b>n/\min(p,q)$. Then*

$$\left\|\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}|\Box_k^* f|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}, \quad (31)$$

$$\left\|\left(\sum_{k_n\in\mathbb{Z}}\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}|\Box_k^* f|^q\right)^{r/q}\right)^{1/r}\right\|_{L^p(\mathbb{R}^n)} \quad (32)$$

are equivalent norms in $W^{p,q}_s(\mathbb{R}^n)$ and $W^{p,q,r}_s(\mathbb{R}^n)$, respectively.

The proof is a direct consequence of Lemma 7, taking $f_k=\Box_k f$. See also [14, Proposition].

3. Proof of the Main Results

First, we narrate the idea of the proof. We give an equivalent formulation for $\Box_{\bar{k}}(\mathbb{T}f)(\bar{x})$, a function in $\mathbb{R}^{n-1}$, via some $\Box_{\bar{k},l}f(\bar{x},0)$, a function in $\mathbb{R}^n$. Then we compute for pointwise estimates between the corresponding ℓ^q norms and $\ell^r_{k_n}\ell^q_{\bar{k}}$ norms for cases $0<q<1$ and $1\leq q<\infty$, separately. Finally, taking $L^p(\mathbb{R}^{n-1})$ norms and using our equivalent norms in Proposition 9, we arrive to our conclusion.

We denote by $\mathcal{F}_{\bar{x}}(\mathcal{F}^{-1}_{\bar{\xi}})$ the partial (inverse) Fourier transform on $\bar{x}(\bar{\xi})\in\mathbb{R}^{n-1}$. Write $\{\varphi_{\bar{k}}\}_{\bar{k}\in\mathbb{Z}^{n-1}}$ as versions of (9) in $\mathbb{R}^{n-1}$. By the support property of $\varphi_{\bar{k}}$, we observe

$$\begin{aligned}\Box_{\bar{k}}(\mathbb{T}f)(\bar{x})&=\left(\mathcal{F}^{-1}_{\bar{\xi}}\varphi_{\bar{k}}\mathcal{F}_{\bar{x}}\right)(\mathbb{T}f)(\bar{x})\\&=\sum_{l\in\mathbb{Z}^n}\left\{\mathcal{F}^{-1}_{\bar{\xi}}\varphi_{\bar{k}}\mathcal{F}_{\bar{x}}\left[\left(\mathcal{F}^{-1}\varphi_l\mathcal{F}f\right)(\bar{y},0)\right]\right\}(\bar{x})\\&=\sum_{l\in\mathbb{Z}^n}\chi_{(|\bar{k}-\bar{l}|\leq 1)}\left(\mathcal{F}^{-1}\psi_{\bar{k},l}\mathcal{F}f\right)(\bar{x},0)\\&=\sum_{l\in\mathbb{Z}^n}\chi_{(|\bar{k}-\bar{l}|\leq 1)}\Box_{\bar{k},l}f(\bar{x},0),\end{aligned} \quad (33)$$

where $\psi_{\bar{k},l}(\xi)=\varphi_{\bar{k}}(\bar{\xi})\varphi_l(\xi)$, $l=(\bar{l},l_n)$, and $\Box_{\bar{k},l}f:=\mathcal{F}^{-1}\psi_{\bar{k},l}\mathcal{F}f$. Note that the left-hand side is a function in $\mathbb{R}^{n-1}$ while the right-hand side is a function in $\mathbb{R}^n$.

Recall our maximal function (30) and take $y_1=y_2=\cdots=y_{n-1}=0$, $y_n=x_n$; we have, for $|x_n|\leq 1$,

$$|\Box_k f(\bar{x},0)|\lesssim\Box_k^* f(x). \quad (34)$$

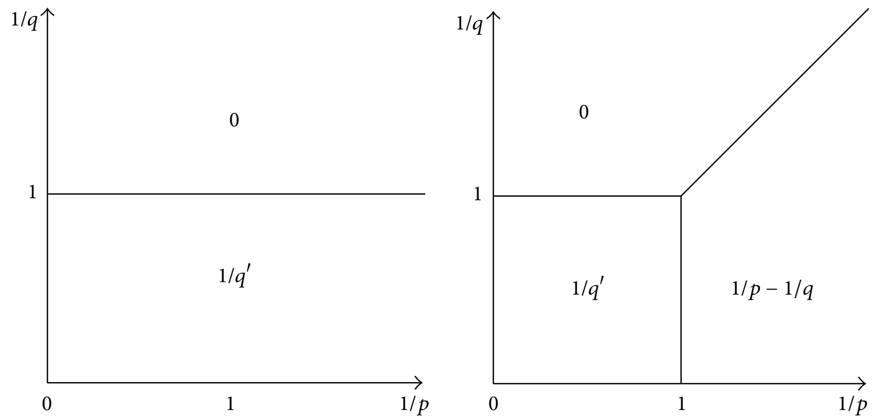
Proof of Theorem 2. We start by taking the ℓ^q-norm of (33). We write

$$\begin{aligned}&\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}|\Box_{\bar{k}}(\mathbb{T}f)(\bar{x})|^q\right)^{1/q}\\&=\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}\left(\sum_{l\in\mathbb{Z}^n}\chi_{(|\bar{k}-\bar{l}|\leq 1)}\Box_{\bar{k},l}f(\bar{x},0)\right)^q\right)^{1/q}.\end{aligned} \quad (35)$$

For $0<q<1$, we estimate (35) by

$$\begin{aligned}&\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{k}\rangle^{sq}|\Box_{\bar{k}}(\mathbb{T}f)(\bar{x})|^q\right)^{1/q}\\&\lesssim\left(\sum_{l\in\mathbb{Z}^n}\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\langle\bar{l}\rangle^{sq}\chi_{(|\bar{k}-\bar{l}|\leq 1)}\left|\Box_{\bar{k},l}f(\bar{x},0)\right|^q\right)^{1/q}\\&=\left(\sum_{l_n\in\mathbb{Z}}\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\langle\bar{l}\rangle^{sq}\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\chi_{(|\bar{k}-\bar{l}|\leq 1)}\left|\Box_{\bar{k},l}f(\bar{x},0)\right|^q\right)^{1/q}.\end{aligned} \quad (36)$$

Note that $\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\chi_{(|\bar{k}-\bar{l}|\leq 1)}|\Box_{\bar{k},l}f(\bar{x},0)|^q=\sum_{j=1}^{n-1}|\Box_{\overline{l\pm e_j},l}f(\bar{x},0)|^q$, where e_j is the jth column of the identity matrix. In the



(a) (b)

FIGURE 1: Comparison between the critical regularity index s for $\mathbb{T}W^{p,q}_{s+\epsilon}(\mathbb{R}^n) = W^{p,q}(\mathbb{R}^{n-1})$ (a) and $\mathbb{T}M^{p,q}_{s+\epsilon}(\mathbb{R}^n) = M^{p,q}(\mathbb{R}^{n-1})$ (b).

sequel, it suffices to consider only the case $j = 1$. Moreover, we write $\widetilde{\square}_l f := \square_{\overline{l\pm e_1},l} f$ for some ψ_l satisfying (9). Using (34) we have

$$\begin{aligned}\left(\sum_{l_n\in\mathbb{Z}}\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\left\langle\bar{l}\right\rangle^{sq}\left|\square_{\overline{l\pm e_1},l} f(\bar{x},0)\right|^q\right)^{1/q} \\ \lesssim \left(\sum_{l_n\in\mathbb{Z}}\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\left\langle\bar{l}\right\rangle^{sq}\left|\widetilde{\square}_l^* f(\bar{x},x_n)\right|^q\right)^{1/q}.\end{aligned} \tag{37}$$

Combining (36) and (37), then taking the $L^p(\mathbb{R}^{n-1})$-norm, and raising to pth power give

$$\begin{aligned}&\left\|f(\bar{x},0)\right\|^p_{W^{p,q}_s(\mathbb{R}^{n-1})} \\ &\lesssim \left\|\left(\sum_{l\in\mathbb{Z}^n}\left\langle\bar{l}\right\rangle^{sq}\widetilde{\square}_l^{*q} f(\bar{x},x_n)\right)^{1/q}\right\|^p_{L^p(\mathbb{R}^{n-1})}.\end{aligned} \tag{38}$$

Integrating over $x_n \in [0,1]$,

$$\begin{aligned}&\left\|f(\bar{x},0)\right\|_{W^{p,q}_s(\mathbb{R}^{n-1})} \\ &\lesssim \left\|\left(\sum_{l\in\mathbb{Z}^n}\left\langle\bar{l}\right\rangle^{sq}\widetilde{\square}_l^{*q} f(\bar{x},x_n)\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\|f\right\|_{W^{p,q,q}_s(\mathbb{R}^n)}.\end{aligned} \tag{39}$$

Note that the last inequality follows from Proposition 9.

For $1 \le q \le \infty$, we use Minkowski's inequality to give an upper bound of (35) as follows:

$$\begin{aligned}&\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\left\langle\bar{k}\right\rangle^{sq}\left|\square_{\bar{k}}(\mathbb{T}f)(\bar{x})\right|^q\right)^{1/q} \\ &\lesssim \left(\sum_{\bar{k},\bar{l}\in\mathbb{Z}^{n-1}}\left\langle\bar{k}\right\rangle^{sq}\left(\sum_{l_n\in\mathbb{Z}}\chi_{(|\bar{k}-\bar{l}|\le 1)}\square_{\bar{k},l} f(\bar{x},0)\right)^q\right)^{1/q} \\ &\lesssim \sum_{l_n\in\mathbb{Z}}\left(\sum_{\bar{k},\bar{l}\in\mathbb{Z}^{n-1}}\left\langle\bar{k}\right\rangle^{sq}\chi_{(|\bar{k}-\bar{l}|\le 1)}\left|\square_{\bar{k},l} f(\bar{x},0)\right|^q\right)^{1/q} \\ &\lesssim \sum_{l_n\in\mathbb{Z}}\left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}}\left\langle\bar{l}\right\rangle^{sq}\left|\widetilde{\square}_l^* f(\bar{x},x_n)\right|^q\right)^{1/q}.\end{aligned} \tag{40}$$

Repeating the arguments above on (40) gives us the estimate

$$\left\|f(\bar{x},0)\right\|_{W^{p,q}_s(\mathbb{R}^{n-1})} \lesssim \left\|f\right\|_{W^{p,q,1}_s(\mathbb{R}^n)}. \tag{41}$$

Hence, we arrive to our desired estimates.

Let $\eta' \in \mathcal{S}(\mathbb{R})$ be a function with $\operatorname{supp}\eta' \subset (-1/4, 1/4)$ and $(\mathcal{F}^{-1}_{\xi_n})\eta'(0) = 1$. For any $f \in W^{p,q}_s(\mathbb{R}^{n-1})$, we define $g(x) = (\mathbb{T}^{-1}f)(x) := [(\mathcal{F}^{-1}_{\xi_n})\eta'(x_n)]f(\bar{x})$. We easily see that $g(\bar{x},0) = f(\bar{x})$ and $\square_k g = 0$ when $|k_n| \ge 3$. Moreover, we can decompose $\square_k g = \square_{\bar{k}} f \cdot \square_{k_n}(\mathcal{F}^{-1}_{\xi_n}\eta')$ due to the way φ_k is defined in (9). Now we do an estimate

$$\begin{aligned}&\left\|g\right\|_{W^{p,q,q\wedge 1}_s(\mathbb{R}^n)} \\ &= \left\|\left(\sum_{k_n\in\mathbb{Z}}\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\left\langle\bar{k}\right\rangle^{sq}\left|\square_k g\right|^q\right)^{q\wedge 1/q}\right)^{1/q\wedge 1}\right\|_{L^p(\mathbb{R}^n)} \\ &= \left\|\left(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\left\langle\bar{k}\right\rangle^{sq}\left|\square_{\bar{k}} f\right|^q\right)^{1/q}\right. \\ &\quad\left.\cdot\left(\sum_{|k_n|\le 2}\left|\square_{k_n}\left(\mathcal{F}^{-1}_{\xi_n}\eta'\right)\right|^{1\wedge q}\right)^{1/q\wedge 1}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\|f\right\|_{W^{p,q}_s(\mathbb{R}^{n-1})}.\end{aligned} \tag{42}$$

Thus, $\mathbb{T}^{-1} : W^{p,q}_s(\mathbb{R}^{n-1}) \to W^{p,q,q\wedge 1}_s(\mathbb{R}^n)$. □

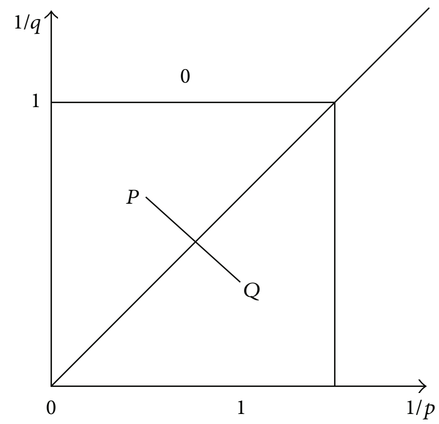


Figure 2: Contradiction argument using interpolation.

As the end of this paper, we discuss the optimality of Corollary 3. We recall the counterexample given in [13]. For $1 < p, q < \infty$, there exists a function which shows

$$\mathbb{T} : M^{p,q}_{1/q'}\left(\mathbb{R}^n\right) \nrightarrow M^{p,q}_0\left(\mathbb{R}^{n-1}\right). \tag{43}$$

Since $M^{q,q} = W^{q,q}$, we also have $\mathbb{T} : W^{q,q}_{1/q'}(\mathbb{R}^n) \nrightarrow W^{q,q}_0(\mathbb{R}^{n-1})$. Hence, Corollary 3 is sharp for $p = q$, $1 < p, q < \infty$ (refer to Figure 1). We now claim that it is also sharp for all $1 < p, q < \infty$. Contrary to our claim, suppose $s = 1/q'$ implies $\mathbb{T}W^{p,q}_s(\mathbb{R}^n) = W^{p,q}(\mathbb{R}^{n-1})$. Then, by interpolation with the estimate for a point $Q(p_1, q_1)$ with $s = 1/q'_1$, one would obtain an improvement for the segment connecting $P(p, q)$ and $Q(p_1, q_1)$ (refer to Figure 2), which is not possible.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Existence of Solutions for Degenerate Elliptic Problems in Weighted Sobolev Space

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This paper is devoted to the study of the existence of solutions to a general elliptic problem $Au + g(x, u, \nabla u) = f - \operatorname{div}F$, with $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^{N} L^{p'}(\Omega, \omega_i^*)$, where A is a Leray-Lions operator from a weighted Sobolev space into its dual and $g(x, s, \xi)$ is a nonlinear term satisfying $g(x, s, \xi)\operatorname{sgn}(s) \geq \rho \sum_{i=1}^{N} \omega_i |\xi_i|^p$, $|s| \geq h > 0$, and a growth condition with respect to ξ. Here, ω_i, ω_i^* are weight functions that will be defined in the Preliminaries.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ and let p be a real number with $1 < p < \infty$. Denote by X the weighted Sobolev space $W_0^{1,p}(\Omega, \omega)$, associated with a vector of weight functions $\omega = \{\omega_i(x)\}_{0 \leq i \leq N}$, which is endowed with the usual norm $\|\cdot\|_{1,p,\omega}$. In this paper, we consider a general class of degenerate elliptic problems:

$$\begin{aligned} Au + g(x, u, \nabla u) &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $Au = -\operatorname{div}(a(x, u, \nabla u))$ and the right-hand side term $\mu = f - \operatorname{div}F$, where $f \in L^1(\Omega)$, $F \in \prod_{i=1}^{N} L^{p'}(\Omega, \omega_i^*)$. We also assume the following:

(H1) The expression

$$|||u|||_X := \left(\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u(x)}{\partial x_i}\right|^p \omega_i(x)\, dx\right)^{1/p} \tag{2}$$

is a norm defined on X and it is equivalent to $\|\cdot\|_{1,p,\omega}$.

(H2) There exist a weight function $\sigma(x)$ and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \tag{3}$$

with $q' = q/(q-1)$. The Hardy inequality

$$\begin{aligned} &\left(\int_\Omega |u(x)|^q \sigma\, dx\right)^{1/q} \\ &\quad \leq C \left(\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u(x)}{\partial x_i}\right|^p \omega_i(x)\, dx\right)^{1/p} \end{aligned} \tag{4}$$

holds for every $u \in X$ with a constant $C > 0$ independent of u. Moreover, the embedding

$$X \hookrightarrow L^q(\Omega, \sigma) \tag{5}$$

is compact. Interested reader may refer to [1] for some examples of weights which satisfy the above Hardy inequality (see (4)).

(**H3**) $a(x,s,\xi) = \{a_i(x,s,\xi)\}_{1\le i\le N} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-valued function, and for all $i = 1,\dots,N$, there hold

$$\left|a_i(x,s,\xi)\right| \le c_1 \omega_i^{1/p}(x) \cdot \left[k(x) + \sigma^{1/p'} |s|^{q/p'} + \sum_{j=1}^{N} \omega_j^{1/p'}(x) \left|\xi_j\right|^{p-1}\right], \tag{6}$$

$$a(x,s,\xi)\cdot\xi \ge c_0 \sum_{i=1}^{N} \omega_i(x) \left|\xi_i\right|^p, \tag{7}$$

$$\left[a(x,s,\xi) - a(x,s,\eta)\right]\cdot(\xi-\eta) > 0, \quad \xi \ne \eta \in \mathbb{R}^N, \tag{8}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$, $1/p + 1/p' = 1$, and the constants c_0, c_1 are both positive.

(**H4**) Let $g(x,s,\xi)$ be a Carathéodory function satisfying the following assumptions:

$$g(x,s,\xi)\,\mathrm{sgn}(s) \ge \rho \sum_{i=1}^{N} \omega_i \left|\xi_i\right|^p, \quad |s| \ge h, \tag{9}$$

for some $h, \rho > 0$, and

$$\left|g(x,s,\xi)\right| \le b(|s|)\left(\sum_{i=1}^{N} \omega_i \left|\xi_i\right|^p + d(x)\right), \tag{10}$$

with $b : \mathbb{R}^+ \to \mathbb{R}^+$, a continuous increasing function, and $d(x)$, a nonnegative function in $L^1(\Omega)$.

In the past decade, much attention has been devoted to nonlinear elliptic equations because of their wide application to physical models such as non-Newtonian fluids, boundary layer phenomena for viscous fluids, and chemical heterogenous model. When $-\mathrm{div}F = 0$, Akdim et al. [2] proved in the variational setting, under assumptions (**H1**)–(**H4**), that, for every $f \in W^{-1,p'}(\Omega,\omega^*)$, with g satisfying the sign condition

$$g(x,s,\xi)\,\mathrm{sgn}(s) \ge 0. \tag{11}$$

Problem (1) has a solution $u \in W_0^{1,p}(\Omega,\omega)$, where the authors used the approach based on the strong convergence of the positive part u_n^+ (negative part u_n^-) of u_n (the approximating sequence of u). Ammar [3] extended this existence result to problems with general data $f \in L^1(\Omega)$, under hypotheses (**H1**)–(**H4**). They also used a similar approach to prove the existence of renormalized solutions. When $-\mathrm{div}F \ne 0$, Aharouch et al. [4] proved the existence result for problem (1), by assuming the sign condition (11). For more details on weighted Sobolev spaces, the readers may refer to [5].

Boccardo et al. [6] considered the nonlinear boundary value problem

$$Au + g(x,u,\nabla u) = \mu, \tag{12}$$

where $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ and $g(x,u,\nabla u) \in L^1(\Omega)$ with sign condition (9) for large values of s. By combining the truncation technique with some delicate test functions, the authors showed that the problem has a solution $u \in W_0^{1,p}(\Omega)$. Mainly motivated by [4, 6], we investigate the elliptic problem (1) in weighted Sobolev space. By choosing test functions different from those employed in [4, 6], we show that problem (1) admits at least one weak solution with (9) instead of the sign condition (11). It is worth pointing out that (9) gives a sign condition on $g(x,s,\xi)$ only for large values of s, which brings about many difficulties. The essential one of those is that we have to construct some new test functions to obtain the a priori estimates of the approximation solutions u_n since the usual one $T_k(u_n)$ is not a proper test function for our problem. The outline of this paper is as follows. In Section 2, we give some preliminaries and some technical lemmas. The main results will be stated and proved in Section 3.

2. Preliminaries

In this section, we give some preliminaries (see [5]). Throughout this section, we assume that the vector field $a(x,s,\xi) = \{a_i(x,s,\xi)\}\ 1 \le i \le N : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies assumptions (6)–(8) and g satisfies (9)-(10). Let $\omega = \{\omega_i(x)\}_{0\le i\le N}$ be a vector of measurable weight functions strictly positive a.e. in Ω, such that

$$\omega_i \in L^1(\Omega), \qquad \omega_i^{-1/(p-1)} \in L^1(\Omega). \tag{13}$$

We define the weighted space with weight γ on Ω as

$$L^p(\Omega,\gamma) = \left\{u = u(x) : u\gamma^{1/p} \in L^p(\Omega)\right\}. \tag{14}$$

With this space, we equip the norm

$$\|u\|_{p,\gamma} = \left(\int_\Omega |u(x)|^p \gamma(x)\,dx\right)^{1/p}. \tag{15}$$

We denote by $W_0^{1,p}(\Omega,\omega)$ the space of all real-valued functions $u \in L^p(\Omega,\omega_0)$ such that the derivatives (see [5]) in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,\omega_i) \quad \forall i = 1,\dots,N, \tag{16}$$

endowed with the norm

$$\|u\|_{1,p,\omega} = \left(\int_\Omega |u(x)|^p \omega_0(x)\,dx + \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u(x)}{\partial x_i}\right|^p \omega_i(x)\,dx\right)^{1/p}. \tag{17}$$

Let $X{:=}W_0^{1,p}(\Omega,\omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p,\omega}$. Then, $(X,\|\cdot\|_{1,p,\omega})$ is a reflexive Banach space whose dual is equivalent to $W^{-1,p'}(\Omega,\omega^*)$, where $\omega^* = \{\omega_i^* = \omega_i^{1-p'} = \omega_i^{-p'/p}\}$, $i = 1,\dots,N$, and $p' = p/(p-1)$. As usual, for

s, k in $\mathbb{R}$, with $k \geq 0$, we denote $T_k(s) = \max(-k, \min(k, s))$ and $G_k(s) = s - T_k(s)$.

The following lemmas will be needed throughout this paper (refer to [2, 7]).

Lemma 1. *Let a and b be two nonnegative real numbers, and let*

$$\varphi(s) = se^{\theta s^2}, \tag{18}$$

with $\theta = b^2/4a^2$. Then,

$$a\varphi'(s) - b|\varphi(s)| \geq \frac{a}{2}, \quad s \in \mathbb{R}. \tag{19}$$

Lemma 2. *Let $g \in L^r(\Omega, \gamma)$ and $g_n \in L^r(\Omega, \gamma)$, with $\|g_n\|_{L^r(\Omega,\gamma)} \leq c$, $1 < r < \infty$. If $g_n \to g$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega, \gamma)$, where γ is a weight function on Ω.*

Lemma 3 (assume (**H1**)). *Let $G : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $G(0) = 0$. Let $u \in W_0^{1,p}(\Omega, \omega)$. Moreover, if the set D of discontinuity points of G' is finite, then*

$$\frac{\partial G \circ u}{x_i} = \begin{cases} 0, & \text{a.e. in } \{x \in \Omega : u(x) \in D\}, \\ G'(u)\dfrac{\partial u}{\partial x_i}, & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}. \end{cases} \tag{20}$$

Lemma 4 (assume (**H1**)). *Let $u \in W_0^{1,p}(\Omega, \omega)$ and $T_k(u)$, $k \in \mathbb{R}^+$, be the usual truncation. Then, $T_k(u) \in W_0^{1,p}(\Omega, \omega)$. Moreover, one has*

$$T_k(u) \longrightarrow u \quad \text{strongly in } W_0^{1,p}(\Omega, \omega). \tag{21}$$

Lemma 5 (assume (**H1**) and (**H2**)). *Let $\{u_n\}$ be a sequence of functions in $W_0^{1,p}(\Omega, \omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, \omega)$ and*

$$\lim_{n\to\infty} \int_\Omega [a(x, u_n, \nabla u_n) - a(x, u, \nabla u)] \cdot \nabla(u_n - u)\, dx = 0. \tag{22}$$

Then, $u_n \to u$ strongly in $W_0^{1,p}(\Omega, \omega)$.

3. Main Results

Firstly, we give the definition of weak solution for problem (1).

Definition 6. One says $u \in W_0^{1,p}(\Omega, \omega)$ is a weak solution to problem (1), provided that

$$\begin{aligned} &\int_\Omega a(x, u, \nabla u) \cdot \nabla v\, dx + \int_\Omega g(x, u, \nabla u)\, v\, dx \\ &\quad = \int_\Omega f v\, dx + \int_\Omega F \cdot \nabla v\, dx, \\ &\qquad \text{for every } v \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega). \end{aligned} \tag{23}$$

Now, we will state and prove our main result on the existence of weak solutions to problem (1).

Theorem 7. *Let f be in $L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$. Then, there exists at least one solution u to problem (1).*

Proof. The proof will be divided into 5 steps.

Step 1 (the approximation equation). We introduce the following approximation equation of problem (1). Let f_n be a sequence of $L^\infty(\Omega)$ functions that converges to f strongly in $L^1(\Omega)$ and let $n \in N$,

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + (1/n)|g(x, s, \xi)|}; \tag{24}$$

then $g_n(x, s, \xi)$ is bounded and satisfies (10) and

$$g_n(x, s, \xi) \cdot \operatorname{sgn}(s) \geq 0, \tag{25}$$

for almost every x in Ω, for every ξ in $\mathbb{R}^N$, and for every s in $\mathbb{R}$ with $|s| \geq h$. By the results of [2], there exists a solution $u_n \in W_0^{1,p}(\Omega, \omega)$ of

$$\begin{aligned} A(u_n) + g_n(x, u_n, \nabla u_n) &= f_n - \operatorname{div} F \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{26}$$

which satisfies

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\, v\, dx \\ &\quad = \int_\Omega f_n v\, dx + \int_\Omega F \cdot \nabla v\, dx, \end{aligned} \tag{27}$$

for every $v \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

Step 2 (the weak convergence $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega)$). Take $v = \varphi(T_h(u_n))$ as a test function in (27), where $h > 0$ is defined in (9) and $\varphi(s)$ is as in (19). Writing $\varphi_h' = \varphi'(T_h(u_n))$ and $\varphi_h = \varphi(T_h(u_n))$ for simplicity, we have

$$\begin{aligned} &\int_\Omega (a(x, u_n, \nabla u_n)) \cdot \nabla T_h(u_n)\, \varphi_h' dx \\ &\quad + \int_\Omega g_n(x, u_n, \nabla u_n)\, \varphi_h dx \\ &\quad = \int_\Omega f_n \varphi_h dx + \int_\Omega F \cdot \nabla T_h(u_n)\, \varphi_h' dx. \end{aligned} \tag{28}$$

Thanks to Young's inequality and (7), we have

$$\begin{aligned} &c_0 \int_\Omega \sum_{i=1}^N \left|\frac{\partial T_h(u_n)}{\partial x_i}\right|^p \omega_i(x)\, \varphi_h' dx \\ &\quad + \int_\Omega g_n(x, u_n, \nabla u_n)\, \varphi_h dx \\ &\quad \leq \varphi(h) \int_\Omega |f_n|\, dx + \varphi'(h) \int_\Omega \sum_{i=1}^N \left|F_i \cdot \omega_i^{-1/p}\right|^{p'} dx \\ &\quad + \frac{c_0}{2} \int_\Omega \sum_{i=1}^N \left|\frac{\partial T_h(u_n)}{\partial x_i}\right|^p \omega_i(x)\, \varphi_h' dx. \end{aligned} \tag{29}$$

Since $\{f_n\}$ is bounded in $L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$, it follows from the above inequality that

$$\frac{c_0}{2}\int_\Omega \sum_{i=1}^N \left|\frac{\partial T_h(u_n)}{\partial x_i}\right|^p \omega_i(x)\varphi_h' dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_h dx \le C, \tag{30}$$

where C is independent of n. Splitting the second term on the left-hand side where $|u_n| < h$ and $|u_n| \ge h$, we can write

$$\begin{aligned}\int_\Omega g_n(x, u_n, \nabla u_n)\varphi_h dx &= \int_{\{|u_n|<h\}} g_n(x, u_n, \nabla u_n)\varphi_h dx \\ &+ \int_{\{|u_n|\ge h\}} g_n(x, u_n, \nabla u_n)\varphi_h dx.\end{aligned} \tag{31}$$

Using (9) and (10), we get

$$\begin{aligned}&\int_{\{|u_n|\ge h\}} g_n(x, u_n, \nabla u_n)\varphi_h dx \ge \rho\varphi(h) \cdot \int_{\{|u_n|\ge h\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i(x)\,dx, \\ &\left|\int_{\{|u_n|<h\}} g_n(x, u_n, \nabla u_n)\varphi_h dx\right| \le b(h) \cdot \left(\int_\Omega \sum_{i=1}^N \left|\frac{\partial T_h(u_n)}{\partial x_i}\right|^p \omega_i(x)\varphi_h dx + \varphi(h)\|d\|_{L^1(\Omega)}\right).\end{aligned} \tag{32}$$

Hence,

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \left|\frac{\partial T_h(u_n)}{\partial x_i}\right|^p \omega_i(x)\left[\frac{c_0}{2}\varphi_h' - b(h)|\varphi_h|\right]dx \\ &+ \rho\varphi(h)\int_{\{|u_n|\ge h\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i(x)\,dx \le C.\end{aligned} \tag{33}$$

Recalling (19) in Lemma 1, let $a = c_0/2$, $b = b(h)$; we then obtain

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial T_h(u_n)}{\partial x_i}\right|^p \omega_i(x)\,dx + \int_{\{|u_n|\ge h\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i(x)\,dx \le C, \tag{34}$$

which implies

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i(x)\,dx \le C, \tag{35}$$

or equivalently

$$\left\||u_n|\right\|_X \le C, \tag{36}$$

where C is some positive constant. Therefore, we can extract a subsequence, still denoted by itself, such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega, \omega). \tag{37}$$

By (5) and (37), we have for a subsequence $u_n \to u$ strongly in $L^q(\Omega, \omega)$ and a.e. in Ω. Then, $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega, \omega)$. Hence, by the results of [8], we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega, \omega). \tag{38}$$

Step 3 (the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega, \omega)$). For every $k \ge h$, we will prove that $T_k(u_n)$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega, \omega)$. We first prove that

$$\lim_{k\to+\infty} \sup_{n\in N^*} \int_{\{|u_n|\ge k\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx = 0. \tag{39}$$

Here, we denote by N^* the set of natural numbers. Choosing $v = T_k(u_n) - T_{k-1}(u_n)$ as a test function in (27) with $k \ge h+1$, using (7) and Young's inequality, we obtain

$$\begin{aligned}&\frac{c_0}{2}\int_\Omega \sum_{i=1}^N \omega_i(x)\left|\frac{\partial(T_k(u_n) - T_{k-1}(u_n))}{\partial x_i}\right|^p dx \\ &+ \int_\Omega g_n(x, u_n, \nabla u_n)(T_k(u_n) - T_{k-1}(u_n))\,dx \\ &\le \int_{\{|u_n|\ge k-1\}} f_n(T_k(u_n) - T_{k-1}(u_n))\,dx \\ &+ \int_{\{k-1\le|u_n|\le k\}} \sum_{i=1}^N \left|F_i \cdot \omega_i^{-1/p}\right|^{p'} dx.\end{aligned} \tag{40}$$

Noticing (9) and that $T_k(u_n) - T_{k-1}(u_n)$ has the same sign as $g_n(x, u_n, \nabla u_n)$ if $|u_n| > h$ and is zero if $|u_n| \le h$, we get

$$\begin{aligned}&g_n(x, u_n, \nabla u_n)(T_k(u_n) - T_{k-1}(u_n)) \\ &\ge |g_n(x, u_n, \nabla u_n)|\,\chi_{\{|u_n|\ge k\}}.\end{aligned} \tag{41}$$

Dropping the nonnegative term, we have

$$\begin{aligned}\int_{\{|u_n|\ge k\}} |g_n(x, u_n, \nabla u_n)|\,dx &\le \int_{\{|u_n|\ge k-1\}} |f_n|\,dx \\ &+ \int_{\{k-1\le|u_n|\le k\}} \sum_{i=1}^N \left|F_i \cdot \omega_i^{-1/p}\right|^{p'} dx.\end{aligned} \tag{42}$$

Since

$$\begin{aligned}\operatorname{meas}\{|u_n| \ge k\} &\le \int_\Omega \frac{|u_n|\,\omega_0^{1/p}}{k}\left(\omega_0^{-1}(x)\right)^{1/p} dx \\ &\le \frac{1}{k}\|u_n\|_{L^p(\Omega,\omega_0)}\left\|\left(\omega_0^{-1}\right)^{1/p}\right\|_{L^{p'}(\Omega)} \\ &\le \frac{1}{k}\|u_n\|_{L^p(\Omega,\omega_0)}\left\|\left(\omega_0^{-1}\right)\right\|_{L^{1/(p-1)}(\Omega)}^{1/p},\end{aligned} \tag{43}$$

we obtain

$$\lim_{k\to+\infty}\sup_{n\in N^*}\operatorname{meas}\left(\left\{\left|u_n\right|\geq k-1\right\}\right)=0. \tag{44}$$

Taking into account the fact that $\{f_n\}$ is compact in $L^1(\Omega)$ and $F\in\prod_{i=1}^N L^{p'}(\Omega,\omega_i^*)$, we deduce that

$$\lim_{k\to+\infty}\sup_{n\in N^*}\int_{\{|u_n|\geq k-1\}}\left|f_n\right|dx+\int_{\{k-1\leq|u_n|\leq k\}}\sum_{i=1}^N\left|F_i\cdot\omega_i^{-1/p}\right|^{p'}dx=0. \tag{45}$$

Hence,

$$\lim_{k\to+\infty}\sup_{n\in N^*}\int_{\{|u_n|\geq k\}}\left|g_n\left(x,u_n,\nabla u_n\right)\right|dx=0. \tag{46}$$

Noticing that $k\geq h$ and (9), this completes the proof of assertion (39).

Let $k\geq h$ be fixed, $0<\varepsilon<k$, and choose $v=\varphi(T_\varepsilon(u_n-T_k(u)))$ as a test function in (27), where $\varphi(s)$ is defined in Lemma 1 (refer to [8–10]). We thus obtain

$$\begin{aligned}&\underbrace{\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla\left(T_\varepsilon\left(u_n-T_k(u)\right)\right)\varphi'\left(T_\varepsilon\left(u_n-T_k(u)\right)\right)dx}_{(A)}\\&+\underbrace{\int_\Omega g_n\left(x,u_n,\nabla u_n\right)\varphi\left(T_\varepsilon\left(u_n-T_k(u)\right)\right)dx}_{(B)}\\&=\underbrace{\int_\Omega f_n\varphi\left(T_\varepsilon\left(u_n-T_k(u)\right)\right)dx}_{(C)}\\&+\underbrace{\int_\Omega F\cdot\nabla\left(T_\varepsilon\left(u_n-T_k(u)\right)\right)\varphi'\left(T_\varepsilon\left(u_n-T_k(u)\right)\right)dx.}_{(D)}\end{aligned} \tag{47}$$

In the following, $\delta(\varepsilon,n)$ represents a quantity which converges to zero as firstly $n\to\infty$ and secondly $\varepsilon\to0$. For convenience, we write

$$\begin{aligned}\varphi'_{\varepsilon,n}&=\varphi'\left(T_\varepsilon\left(u_n-T_k(u)\right)\right),\\\varphi_{\varepsilon,n}&=\varphi\left(T_\varepsilon\left(u_n-T_k(u)\right)\right).\end{aligned} \tag{48}$$

Observe that, in the weak* topology of $L^\infty(\Omega)$ and almost everywhere in Ω, we have

$$\begin{aligned}\lim_{\varepsilon\to0}\lim_{n\to\infty}\varphi_{\varepsilon,n}&=0,\\\lim_{\varepsilon\to0}\lim_{n\to\infty}\varphi'_{\varepsilon,n}&=1.\end{aligned} \tag{49}$$

Now, as $\{f_n\}$ is compact in $L^1(\Omega)$ and (49), we have

$$(C)=\delta(\varepsilon,n). \tag{50}$$

Thanks to $T_{k+\varepsilon}(u_n)\rightharpoonup T_k(u)$ *weakly in* $W_0^{1,p}(\Omega,\omega)$, $F\in\prod_{i=1}^N L^{p'}(\Omega,\omega_i^*)$, and (49), we obtain

$$\begin{aligned}(D)&=\int_\Omega F\\&\quad\cdot\nabla\left(T_{k+\varepsilon}\left(u_n\right)-T_k(u)\right)\chi_\varepsilon\left\{\left|u_n-T_k(u)\right|\right\}\varphi'_{\varepsilon,n}dx\\&=\delta(\varepsilon,n),\end{aligned} \tag{51}$$

where $\chi_\varepsilon\{|u_n-T_k(u)|\}=\chi\{|u_n-T_k(u)|\leq\varepsilon\}$. We can decompose (A) as

$$\begin{aligned}&\underbrace{\int_\Omega a\left(x,T_{k+\varepsilon}\left(u_n\right),\nabla T_{k+\varepsilon}\left(u_n\right)\right)\cdot\nabla\left(T_\varepsilon\left(u_n-T_k(u)\right)\right)\varphi'_{\varepsilon,n}dx}_{(E)}\\&+\underbrace{\int_\Omega a\left(x,u_n,\nabla G_{k+\varepsilon}\left(u_n\right)\right)\chi\left\{\left|u_n\right|\geq k+\varepsilon\right\}\nabla\left(T_\varepsilon\left(u_n-T_k(u)\right)\right)\varphi'_{\varepsilon,n}dx.}_{(F)}\end{aligned} \tag{52}$$

Owing to $\{|u_n-T_k(u)|\leq\varepsilon\}\subset\{|u_n|\leq k+\varepsilon\}$, we get

$$\begin{aligned}(F)&=\int_\Omega a\left(x,u_n,\nabla G_{k+\varepsilon}\left(u_n\right)\right)\chi\left\{\left|u_n\right|\geq k+\varepsilon\right\}\\&\quad\cdot\nabla\left(T_{k+\varepsilon}\left(u_n\right)-T_k(u)\right)\chi_\varepsilon\left\{\left|u_n-T_k(u)\right|\right\}\varphi'_{\varepsilon,n}dx.\end{aligned} \tag{53}$$

Since $\nabla T_{k+\varepsilon}(u_n)$ is zero whenever $\nabla G_{k+\varepsilon}(u_n)$ is not zero, hence,

$$\begin{aligned}(F)&=-\int_\Omega a\left(x,u_n,\nabla G_{k+\varepsilon}\left(u_n\right)\right)\\&\quad\cdot\nabla T_k(u)\,\chi\left\{\left|u_n\right|>k+\varepsilon\right\}\chi_\varepsilon\left\{\left|u_n-T_k(u)\right|\right\}\varphi'_{\varepsilon,n}dx.\end{aligned} \tag{54}$$

Since $\nabla T_k(u)\equiv0$ on the set $\{|u|>k+\varepsilon\}$, we see that, as $n\to\infty$,

$$\nabla T_k(u)\,\chi_{\{|u_n|>k+\varepsilon\}}\longrightarrow0,\quad\text{a.e. in }\Omega. \tag{55}$$

As $\nabla T_k(u)\in\prod_{i=1}^N L^p(\Omega,\omega_i)$, Lebesgue's dominated convergence theorem guarantees that

$$\nabla T_k(u)\,\chi_{\{|u_n|>k+\varepsilon\}}\longrightarrow0\quad\text{strongly in }\prod_{i=1}^N L^p\left(\Omega,\omega_i\right),\ n\longrightarrow\infty. \tag{56}$$

By (6), (49), and the fact that $a(x, u_n, \nabla G_{k+\varepsilon}(u_n))$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega, \omega_i^*)$, we obtain

$$(F) = \delta(\varepsilon, n). \quad (57)$$

Now we split (E) into

$$\underbrace{\int_\Omega \left[a\left(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)\right) - a\left(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)\right)\right] \cdot \nabla\left(T_{k+\varepsilon}(u_n) - T_k(u)\right) \varphi'_{\varepsilon,n} dx}_{(G)} + \underbrace{\int_\Omega a\left(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)\right) \cdot \nabla\left(T_{k+\varepsilon}(u_n) - T_k(u)\right) \varphi'_{\varepsilon,n} dx.}_{(H)} \quad (58)$$

We will prove that

$$a_i\left(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)\right) \longrightarrow a_i\left(x, T_{k+\varepsilon}(u), \nabla T_k(u)\right) \quad \text{strongly in } L^{p'}(\Omega, \omega_i^*). \quad (59)$$

In fact,

$$\begin{aligned}
&\left|a_i\left(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)\right)\right|^{p'} \omega_i^{-p'/p} \le c_1 \Bigg[k(x) + \sigma^{1/p'} \left|T_{k+\varepsilon}(u_n)\right|^{q/p'} \\
&\quad + \sum_{i=1}^{N} \omega_i^{1/p'}(x) \left|\frac{\partial T_k(u)}{\partial x_i}\right|^{p-1}\Bigg]^{p'} \le c_2 \Bigg[k(x)^{p'} + \sigma\left|T_{k+\varepsilon}(u_n)\right|^q + \sum_{i=1}^{N} \omega_i(x) \left|\frac{\partial T_k(u)}{\partial x_i}\right|^p\Bigg],
\end{aligned} \quad (60)$$

where c_1, c_2 are positive constants. Since $T_{k+\varepsilon}(u_n) \rightharpoonup T_{k+\varepsilon}(u)$ weakly in $W_0^{1,p}(\Omega, \omega)$ and $W_0^{1,p}(\Omega, \omega) \hookrightarrow L^q(\Omega, \sigma)$ is compact, then $T_{k+\varepsilon}(u_n) \to T_{k+\varepsilon}(u)$ strongly in $L^q(\Omega, \sigma)$ and a.e. in Ω. Hence,

$$\begin{aligned}
&\left|a_i\left(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)\right)\right|^{p'} \omega_i^* \longrightarrow \left|a_i\left(x, T_{k+\varepsilon}(u), \nabla T_k(u)\right)\right|^{p'} \omega_i^* \quad \text{a.e. in } \Omega, \\
&c_2 \left[k(x)^{p'} + \sigma\left|T_{k+\varepsilon}(u_n)\right|^q + \sum_{i=1}^{N} \omega_i(x) \left|\frac{\partial T_k(u)}{\partial x_i}\right|^p\right] \\
&\longrightarrow c_2 \left[k(x)^{p'} + \sigma\left|T_{k+\varepsilon}(u)\right|^q + \sum_{o=1}^{N} \omega_i(x) \left|\frac{\partial T_k(u)}{\partial x_i}\right|^p\right] \quad \text{strongly in } L^q(\Omega, \sigma).
\end{aligned} \quad (61)$$

Then, by the generalized Lebesgue dominated convergence theorem, we deduce (59). By $\partial T_{k+\varepsilon}(u_n)/\partial x_i \rightharpoonup \partial T_k(u)/\partial x_i$ weakly in $L^p(\Omega, \omega_i)$ and (49), we have

$$(H) = \delta(\varepsilon, n). \quad (62)$$

Using (57) and (62), we have

$$\begin{aligned}
(A) = \int_\Omega &\left[a\left(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)\right) - a\left(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)\right)\right] \cdot \nabla\left(T_{k+\varepsilon}(u_n) - T_k(u)\right) \varphi'_{\varepsilon,n} dx + \delta(\varepsilon, n).
\end{aligned} \quad (63)$$

As for the term (B), we decompose it as

$$\underbrace{\int_{\{|u_n| \ge k+\varepsilon\}} g_n(x, u_n, \nabla u_n) \varphi_{\varepsilon,n} dx}_{(I)} + \underbrace{\int_{\{|u_n| < k+\varepsilon\}} g_n(x, u_n, \nabla u_n) \varphi_{\varepsilon,n} dx.}_{(J)} \quad (64)$$

It is clear that on the set $\{u_n \ge k + \varepsilon\}$ we get

$$\varphi_{\varepsilon,n} = \varphi\left(T_\varepsilon\left(u_n - T_k(u)\right)\right) = \varphi(\varepsilon) \ge 0, \quad (65)$$

while on the set $\{u_n < -k - \varepsilon\}$ we get

$$\varphi_{\varepsilon,n} = \varphi\left(T_\varepsilon\left(u_n - T_k(u)\right)\right) = \varphi(-\varepsilon) \le 0. \quad (66)$$

By (9) and the fact that $k \ge h$, we obtain

$$(I) \ge 0. \quad (67)$$

Using (7) and (10) and noticing that $\varepsilon < k$, we have

$$\begin{aligned}
|(J)| &\le \int_{\{|u_n| \le k+\varepsilon\}} b(2k) \left[d(x) + \sum_{i=1}^{N} \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i(x)\right] \cdot \left|\varphi_{\varepsilon,n}\right| dx \le b(2k) \int_{\{|u_n| \le k+\varepsilon\}} d(x) \left|\varphi_{\varepsilon,n}\right| dx \\
&\quad + \frac{b(2k)}{c_0} \int_{\{|u_n| \le k+\varepsilon\}} a(x, u_n, \nabla u_n) \nabla u_n \left|\varphi_{\varepsilon,n}\right| dx \\
&= \delta(\varepsilon, n) + \frac{b(2k)}{c_0} \int_\Omega a\left(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)\right) \cdot \nabla T_{k+\varepsilon}(u_n) \left|\varphi_{\varepsilon,n}\right| dx \le \frac{b(2k)}{c_0} \\
&\quad \cdot \int_\Omega \left[a\left(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)\right)\right.
\end{aligned}$$

$$- a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_k\left(u\right)\right)\right] \cdot \nabla\left(T_{k+\varepsilon}\left(u_n\right)\right.$$
$$\left. - T_k\left(u\right)\right)\left|\varphi_{\varepsilon,n}\right| dx + \int_\Omega a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_k\left(u\right)\right)$$
$$\cdot \nabla\left(T_{k+\varepsilon}\left(u_n\right) - T_k\left(u\right)\right)\left|\varphi_{\varepsilon,n}\right| dx$$
$$+ \int_\Omega a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_{k+\varepsilon}\left(u_n\right)\right) \cdot \nabla T_k\left(u\right)$$
$$\cdot \left|\varphi_{\varepsilon,n}\right| dx + \delta\left(\varepsilon, n\right). \tag{68}$$

By the weak convergence of $\partial T_{k+\varepsilon}(u_n)/\partial x_i \rightharpoonup \partial T_k(u)/\partial x_i$ in $L^p(\Omega, \omega_i)$, (49), and (59), we have

$$\int_\Omega a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_k\left(u\right)\right) \cdot \nabla\left(T_{k+\varepsilon}\left(u_n\right) - T_k\left(u\right)\right)\left|\varphi_{\varepsilon,n}\right| dx = \delta\left(\varepsilon, n\right). \tag{69}$$

Since $a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n))$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$ and (49),

$$\int_\Omega a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_{k+\varepsilon}\left(u_n\right)\right) \cdot \nabla T_k\left(u\right)\left|\varphi_{\varepsilon,n}\right| dx = \delta\left(\varepsilon, n\right). \tag{70}$$

We have

$$|(J)| \le \frac{b\left(2k\right)}{c_0} \int_\Omega \left[a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_{k+\varepsilon}\left(u_n\right)\right) - a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_k\left(u\right)\right)\right] \cdot \nabla\left(T_{k+\varepsilon}\left(u_n\right) - T_k\left(u\right)\right)\left|\varphi_{\varepsilon,n}\right| dx + \delta\left(\varepsilon, n\right). \tag{71}$$

Invoking (63), we have

$$\int_\Omega \left[a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_{k+\varepsilon}\left(u_n\right)\right) - a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_k\left(u\right)\right)\right] \cdot \nabla\left(T_{k+\varepsilon}\left(u_n\right) - T_k\left(u\right)\right)\left[\varphi'_{\varepsilon,n} - \frac{b\left(2k\right)}{c_0}\left|\varphi_{\varepsilon,n}\right|\right] dx = \delta\left(\varepsilon, n\right). \tag{72}$$

Consequently, by (19) and letting $a = 1$ and $b = b(2k)/c_0$, it yields

$$\int_\Omega \left[a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_{k+\varepsilon}\left(u_n\right)\right) - a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_k\left(u\right)\right)\right] \cdot \nabla\left(T_{k+\varepsilon}\left(u_n\right) - T_k\left(u\right)\right) dx = \delta\left(\varepsilon, n\right). \tag{73}$$

Since $\{|u_n| \le k\} \subset \{|u_n| \le k + \varepsilon\}$, we have

$$\begin{aligned}
&\int_\Omega \left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k\left(u\right)\right)\right] \cdot \nabla\left(T_k\left(u_n\right) - T_k\left(u\right)\right) dx \\
&= \int_{\{|u_n|\le k\}} \left[a\left(x, u_n, \nabla u_n\right) - a\left(x, u_n, \nabla T_k\left(u\right)\right)\right] \cdot \nabla\left(u_n - T_k\left(u\right)\right) dx \\
&\le \int_{\{|u_n|\le k+\varepsilon\}} \left[a\left(x, u_n, \nabla u_n\right) - a\left(x, u_n, \nabla T_k\left(u\right)\right)\right] \cdot \nabla\left(u_n - T_k\left(u\right)\right) dx \\
&= \int_\Omega \left[a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_{k+\varepsilon}\left(u_n\right)\right) - a\left(x, T_{k+\varepsilon}\left(u_n\right), \nabla T_k\left(u\right)\right)\right] \cdot \nabla\left(T_{k+\varepsilon}\left(u_n\right) - T_k\left(u\right)\right) dx = \delta\left(\varepsilon, n\right).
\end{aligned} \tag{74}$$

Together with Lemma 5 and the assumptions on a, we obtain

$$T_k\left(u_n\right) \longrightarrow T_k\left(u\right) \quad \text{strongly in } W_0^{1,p}\left(\Omega, \omega\right), \tag{75}$$

which in turn implies

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega. \tag{76}$$

For any measurable set E of Ω, we have

$$\int_E \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx = \int_{E\cap\{|u_n|<k\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx + \int_{E\cap\{|u_n|\ge k\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx. \tag{77}$$

Let $\varepsilon > 0$. Thanks to

$$\int_{E\cap\{|u_n|\ge k\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx \le \int_{\{|u_n|\ge k\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx, \tag{78}$$

by (39), there exists $k \ge h$ such that

$$\int_{E\cap\{|u_n|\ge k\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx \le \frac{\varepsilon}{2}, \quad n \in N^*. \tag{79}$$

While k is fixed, we get

$$\int_{E\cap\{|u_n|<k\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx \le \int_E \sum_{i=1}^N \left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|^p \omega_i dx. \tag{80}$$

Owing to the strong compactness of $\{T_k(u_n)\}$ in $W_0^{1,p}(\Omega, \omega)$, there exists $\delta' > 0$ such that if $\operatorname{meas}(E) < \delta'$, then

$$\int_{E\cap\{|u_n|<k\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p \omega_i dx \le \frac{\varepsilon}{2}, \quad n \in N^*. \tag{81}$$

Hence,

$$\int_E \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx \le \varepsilon, \quad n \in N^*. \tag{82}$$

Thus, the sequence $\{|\nabla u_n|^p\}$ is equi-integrable. Thanks to Vitali theorem, the equi-integrability together with (76) implies that u_n converges strongly to u in $W_0^{1,p}(\Omega, \omega)$.

Step 4 (the strong convergence $g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ in $L^1(\Omega)$). Note that (76) implies that

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \text{ a.e. in } \Omega. \tag{83}$$

On the other hand, for any measurable set E of Ω, we have

$$\begin{aligned} &\int_E |g_n(x, u_n, \nabla u_n)| \, dx \\ &\quad = \int_{E\cap\{|u_n|<k\}} |g_n(x, u_n, \nabla u_n)| \, dx \\ &\quad\quad + \int_{E\cap\{|u_n|\ge k\}} |g_n(x, u_n, \nabla u_n)| \, dx. \end{aligned} \tag{84}$$

Let $\varepsilon > 0$ be fixed. We have

$$\begin{aligned} &\int_{E\cap\{|u_n|\ge k\}} |g_n(x, u_n, \nabla u_n)| \, dx \\ &\quad \le \int_{\{|u_n|\ge k\}} |g_n(x, u_n, \nabla u_n)| \, dx. \end{aligned} \tag{85}$$

Choose $k \ge h$ in (46) such that

$$\int_{E\cap\{|u_n|\ge k\}} |g_n(x, u_n, \nabla u_n)| \, dx \le \frac{\varepsilon}{2}, \quad n \in N^*. \tag{86}$$

By (10), we have

$$\begin{aligned} &\int_{E\cap\{|u_n|<k\}} |g_n(x, T_k(u_n), \nabla T_k(u_n))| \, dx \\ &\quad \le b(k) \int_E \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p \omega_i(x) \, dx + d(x). \end{aligned} \tag{87}$$

Since $d(x)$ belongs to $L^1(\Omega)$ and $T_k(u_n)$ is compact in $W_0^{1,p}(\Omega, \omega)$, there exists $\delta'' > 0$ such that if meas$(E) < \delta''$, then

$$\begin{aligned} &\int_{E\cap\{|u_n|<k\}} |g_n(x, T_k(u_n), \nabla T_k(u_n))| \, dx \le \frac{\varepsilon}{2}, \\ &\qquad n \in N^*. \end{aligned} \tag{88}$$

Thus, we have proved that $\{g_n(x, u_n, \nabla u_n)\}$ is equi-integrable. Invoking (86) and (88) and by Vitali theorem,

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \tag{89}$$

Step 5 (passing to the limit). Now, by passing to the limit in (27), we obtain

$$\begin{aligned} &\int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega g(x, u, \nabla u) \, v \, dx \\ &\quad = \int_\Omega f v \, dx + \int_\Omega F \cdot \nabla v \, dx; \end{aligned} \tag{90}$$

that is, u is a weak solution to problem (1). The proof is complete. □

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally to the paper and read and approved its final version.

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