A REPRESENTATION FORMULA RELATED TO SCHRÖDINGER OPERATORS

SHIJUN ZHENG

Abstract. Let $H = -d^2/dx^2 + V$ be a Schrödinger operator on the real line, where $V \in L^1 \cap L^2$. We define the perturbed Fourier transform $\mathcal{F}$ for H and show that $\mathcal{F}$ is an isometry from the absolute continuous subspace onto $L^2(\mathbb{R})$. This property allows us to construct a kernel formula for the spectral operator $\varphi(H)$. The main theorem improves the previous reslut obtained by the author for certain short-range potentials.

Schrödinger operator is a central subject in the mathematical study of quantum mechanics. Consider the Schrödinger operator $H = -\triangle + V$ on $\mathbb{R}$, where $\triangle = d^2/dx^2$ and the potential function V is real valued. In Fourier analysis, it is well-known that a square integrable function admits an expansion with exponentials as eigenfunctions of $-\triangle$. A natural conjecture is that an L^2 function admits a similar expansion in terms of "eigenfunctions" of H, a perturbation of the Laplacian (see [7]. Ch.XI and the notes), under certain condition on V.

The three dimension analogue was proven true by T.Ikebe [6], a member of Kato's school, in 1960. Later his result was extended by Thor to the higher dimension case [10]. In one dimension, recent related results can be found in e.g., Guerin-Holschneider [5], Christ-Kiselev [4] and Benedetto-Zheng [3].

Throughout this paper we assume $V : \mathbb{R} \to \mathbb{R}$ is in $L^1 \cap L^2$. We shall prove a one-dimensional version of Ikebe's theorem for L^2 functions (Theorem 1). Theorem 2 presents an integral formula for the kernel of the spectral operator $\varphi(H)$ for a continuous function φ with compact support. In a sequel to this paper we shall use this explicit formula to study function spaces associated with H (see [3]).

The *generalized eigenfunctions* $e(x, \xi)$, $\xi \in \mathbb{R}$ of H satisfy

$$(-d^2/dx^2 + V(x))e(x, \xi) = \xi^2 e(x, \xi) \tag{1}$$

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in the sense of distributions.
Definition. The *perturbed Fourier transform* $\mathcal{F}$ on L^2 is given by

$$\begin{aligned}
\mathcal{F}f(\xi) &= \text{l.i.m.}(2\pi)^{-1/2}\int f(x)\overline{e(x,\xi)}\,dx \qquad (2)\\
&= \lim_{N\to\infty}(2\pi)^{-1/2}\int_{-N}^{N} f(x)\overline{e(x,\xi)}\,dx,
\end{aligned}$$

where the convergence is in L^2 norm as $N \to \infty$. By Theorem 1, $\mathcal{F}$ is a well-defined isometry from $\mathcal{H}_{ac}$ onto L^2.

Theorem 1. *Suppose $V \in L^1 \cap L^2$. Then there exists a family of solutions $e(x,\xi)$, $|\xi| \in [0,\infty) \setminus \mathcal{E}_0$, $\mathcal{E}_0$ being a bounded closed set of measure zero, to equation (1) with the following properties.*

(i) If $f \in L^2$, then there exists an element $\tilde{f} \in L^2$ such that

$$\mathcal{F}f(\xi) = \tilde{f}(\xi) \qquad \text{in } L^2.$$

(ii) The adjoint operator $\mathcal{F}^*$ is given by

$$\mathcal{F}^* g = l.i.m._{N\to\infty}\sum_{i=1}^{N}(2\pi)^{-1/2}\int_{\alpha_i \le \xi^2 \le \beta_i} g(\xi)e(x,\xi)\,d\xi$$

in L^2, where $[\alpha_i,\beta_i) \subset (0,\infty)$ are a countable collection of disjoint intervals with $[0,\infty) \setminus \mathcal{E}_0^2$ equal to $\cup_i[\alpha_i,\beta_i)$.

(iii) If $f \in L^2$, then $\|P_{ac}f\|_{L^2} = \|\tilde{f}\|_{L^2}$, where P_{ac} is the projection onto $\mathcal{H}_{ac}$, the absolute continuous subspace in L^2.

(iv) $\mathcal{F} : L^2 \to L^2$ is a surjection. Moreover, $\mathcal{F}\mathcal{F}^* = Id$ and $\mathcal{F}^*\mathcal{F} = P_{ac}$.

(v) If $f \in \mathcal{D}(H)$, then $(Hf)^{\sim}(\xi) = \xi^2\tilde{f}(\xi)$ in L^2.

Remark 1. The proof is based on the ideas of [6] for $3D$. We also use some simplifications as found in Reed and Simon([7]) and Simon [8].

Remark 2. If $|e(x,\xi)| \le C$ a.e. $(x,\xi) \in \mathbb{R}^2$, then we have a "better-looking" form in (ii) of the theorem

$$\mathcal{F}^* g = \text{l.i.m.}(2\pi)^{-1/2}\int g(\xi)e(x,\xi)\,d\xi.$$

If $H = \int \lambda dE_\lambda$ is the spectral resolution of H, define the spectral operator $\varphi(H) := \int \varphi(\lambda)dE_\lambda$ by functional calculus. We prove a representation formula for the integral kernel of $\varphi(H)$.

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis in $\mathcal{H}_p$, the subspace of eigenfunctions in L^2 for H and let λ_k be the eigenvalue corresponding to e_k.

Theorem 2. *Let the operator H be as in Theorem 1. Suppose $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous and has a compact support disjoint from $\mathcal{E}_0^2 := \{\eta^2 : \eta \in \mathcal{E}_0\}$. Then for $f \in L^1 \cap L^2$*

$$\varphi(H)f(x) = \int_{-\infty}^{\infty} K(x,y)f(y)\,dy \tag{3}$$

where $K = K_{ac} + K_p$,

$$K_{ac}(x,y) = (2\pi)^{-1}\int_{-\infty}^{\infty} \varphi(\xi^2)e(x,\xi)\overline{e(y,\xi)}\,d\xi.$$

and

$$K_p(x,y) = \sum_k \varphi(\lambda_k)e_k(x)\bar{e}_k(y).$$

Remark 1. If $|e(x,\xi)| \le C$, a.e. $(x,\xi) \in \mathbb{R}^2$, then, under the same condition the integral expression (3) is valid for any $\varphi \in C(\mathbb{R})$ with compact support.

Remark 2. When φ is smooth with rapid decay and V is compactly supported in $\mathbb{R}^3$, a formula of this type appeared in [9] by Tao.

References

[1] P. Alsholm, G. Schmidt, Spectral and scattering theory for Schrödinger operators, *Arch. Rational Mech. Anal.* **40** (1971), 281–311.

[2] J. J. Benedetto, *Harmonic Analysis and Applications*, CRC Press, Inc., Boca Raton. FL, 1997.

[3] J. J. Benedetto and S. Zheng, Besov spaces for the Schrödinger operator with barrier potential, *submitted*.

[4] M. Christ and A. Kiselev, One-Dimensional Schrödinger operators with slowly decaying potentials: spectra and asymptotics, or, *Baby Fourier Analysis Meets Toy Quantum Mechanics*, Notes for IPAM tutorial, 2001 Workshop on Oscillatory Integrals and Dispersive Equations.

[5] C.-A. Guerin, M. Holschneider, Time-dependent scattering on fractal measures, *J. Math. Physics* **39**(8), 1998.

[6] T. Ikebe, Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory, *Arch. Rational Mech. Anal.* **5** (1960), 1–34. (Erratum, Remarks on the orthogonality of eigenfunctions for the Schrödinger operator on $\mathbb{R}^n$, J. Fac. Sci. Univ. Tokyo Sect.I **17**, 1970)

[7] M. Reed and B. Simon, *Methods of Modern Mathematical Physics III: Scattering Theory*, Academic Press, New York, 1979.

[8] B. Simon, *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms*, Princeton University Press, Princeton, New Jersey, 1971.

[9] T. Tao, Scattering for the $3D$ Schrödinger equation with compactly supported potential, *Preprint*.

[10] D. Thoe, Eigenfunction expansions associated with Schrödinger operators in $\mathbb{R}^n$, $n \geq 4$, *Arch. Rational Mech. Anal.* **26** (1967), 335–356.

[11] Q. Zhang, Global bounds of Schrödinger heat kernels with negative potentials, *J. Func. Anal.* **182**(2001), no.2, 344-370.

[12] S. Zheng, *Besov spaces for Schrödinger operators*, Dissertation, University of Maryland, 2003.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803

E-mail address: szheng@math.lsu.edu

URL: http://www.math.lsu.edu/~szheng