RIESZ TYPE KERNELS OVER THE RING OF INTEGERS OF A LOCAL FIELD

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ABSTRACT Let O denote the ring of integers of a local field. In this note we prove an approximation theorem for the Riesz type kernels $\{\gamma_{\mu,\lambda,n}\}_{n=1}^{\infty}$ over O. The proof requires a sharp estimate of the *Dirichlet kernel* $D_n(x)$ on $P^{-1}\backslash O$, which may also have independent interest. As a conseqence we solve the local field analog of Fine's problem proposed in [1] concerning rates of convergence of the *Fejér sum* on Walsh system . Our approach has potential applications to other operators such as Cesáro means and Abel-Poisson means of "type I".

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1. Introduction

Throughout this paper K will denote a local field, i.e., a locally compact, non-discrete, totally disconnected and complete field. A survey of current developments in the area of harmonic analysis and approximation theory on K can be found in [9]. For general references, we refer to [10] and [13]. Such fields have been completely determined. If K is of finite characteristic, it is a field of formal power series over a finite field. If K is of characteristic zero, it is a p-adic field or a finite algebraic extension of such a field([10], [3]).

As in [10], write $O = \{x \in K : |x| \le 1\}$ and $P^k = \{x \in K : |x| \le q^{-k}\}, k \in Z$. Φ_k is the characteristic function of P^k. Since K is ultrametric , O is the unique maximal compact and open subring in K, and $P = (\beta)$ is the unique maximal ideal in O, where β is the prime element.

Let $\{u(n)\}_{n=0}^{\infty}$ be a complete list of (distinct) coset representatives of O in K^+. For each $k \in Z$, set $v(n) = \beta^{-k}u(n)$, $n \in Z^+ = \{0, 1, 2, \ldots\}$, then $\{\chi_{v(n)}|_{P^k}\}_{n=0}^{\infty}$ constitutes a complete set of characters on P^k . We may denote this relation by $(P^k) = K/P^{-k}$. In particular, write $\chi_n = \chi_{u(n)}$, then $\{\chi_n|_O\}_{n=0}^{\infty}$ is a complete orthonormal system on O. For $f \in L^1(O)$, the Fourier series of f is by convention

$$f(x) \sim \sum_{k=0}^{\infty} c_k \chi_k(x),$$

where

$$c_k = \int_O f(x)\bar{\chi}_k(x)dx. \tag{1}$$

In his fundamental paper [1; §7] on Walsh systems, Fine proposed the following question: If $f \in Lip\,\alpha, 0 < \alpha < 1$ and if $\sigma_n f$ denotes the Fejér means of the Walsh-Fourier series of f, then does

$$\sigma_n f(x) - f(x) = O(n^{-\alpha})\ ? \tag{$*$}$$

The question was answered in Yano[11] by some suitable estimates for the Fejér kernel on dyadic intervals in (0,1). Later on, extensions of this problem have been studied for Cesáro means or on certain compact Vilenkin groups(see [8; p.521] for some historical comments). In this note we consider the similar question for the more general Riesz means on O in local field settings. We confirm that $(*)$ is true for Fejér means on any local field. The critical case $\alpha = 1$ is also included . This is done by sharpening the estimate in [10;p.101] for $\int_{|x|=q} D_n(x)dx$ from $O(n^{\frac{1}{2}})$ to $O((\log n)^{Z(K)})$.

For $\mu > 0, \lambda > 0$ and $n \in N = \{1, 2, 3, \ldots\}$, define the *Riesz kernels of order* (μ, λ) *over* O to be

$$\gamma_{\mu,\lambda,n}(x) = \sum_{k=0}^{n-1}(1 - (\frac{k}{n})^{\mu})^{\lambda}\chi_k(x). \tag{2}$$

Also, the *Riesz means of order* (μ, λ) *of* $f \in L^1(O)$ are the convolutions

$$R_{\mu,\lambda,n}f(x) = \gamma_{\mu,\lambda,n} * f(x) = \sum_{k=0}^{n-1}(1 - (\frac{k}{n})^{\mu})^{\lambda}c_k\chi_k(x), \tag{3}$$

where c_k is the $k-$th Fourier coefficient of f in (1). Note that if $\lambda = 0, R_{\mu,\lambda,n}f(x)$ equals the $n-$th partial sum of the Fourier series of f, and $\gamma_{\mu,\lambda,n}f(x)$ is the *Dirichlet kernel* $D_n = \sum_{k=0}^{n-1}\chi_k$. Moreover, if $\lambda = \mu = 1$, (3) is just the *n-th Fejér sum.*

In §2, based on the estimate for the Dirichlet kernel(Theorem 1) we obtain the estimates for the Riesz kernels $\gamma_{\mu,\lambda,n}(x)$, from which follows quickly the approximation theorem for the Riesz means. In §3, we study another type of Riesz kernels, which seem to arise more naturally on local fields. The associated operators are closely connected with the concept of fractional derivatives on K. The *Riesz kernel of order* μ *and of type I* is defined as

$$\tilde{\gamma}_{\mu,n}(x) = \sum_{k=0}^{n-1}(1 - |u(k)u(n)^{-1}|^{\mu})\chi_k(x). \tag{4}$$

With this type of Riesz kernels a more precise and higher degree of approximation for Lipschitz functions becomes possible (Theorem 3). We mention that although we do not actually need it in estimating $\tilde{\gamma}_{\mu,n}$, Theorem 1 is particularly useful in treating some more difficult summability methods such as the Riesz means defined in (3). Finally, after some appropriate relationship between $Lip(\alpha; X)$ and $\mathcal{D}(D_X^{\alpha})$ is given, we obtain, specifically for $\tilde{R}_{\mu,n}f$, the analog of Leindler's result on ordinary Riesz means on the circle group [6].

2. The Main Results

The resources come from [12] and [10]. From [12] we exploit the explicit formulas of characters for finite algebraic extensions of a p-series field and a p-adic field, respectively. The estimates of $D_n(x)$ on $P^{-1}\backslash O$ are hence obtained. Then by applying Theorem 1, a delicate modification of the method used in [10; Ch.II] gives the desired estimates for Riesz kernels $\gamma_{\mu,\lambda,n}$ (Lemma 6), hence Theorem 2.

Using the natural lexicographic ordering for u(n) as in [10], we have the following basic decomposition of χ_n.

Lemma 1. ([10; p.85]). For all $r \in Z^+, k \in\ Z^+, 0 \le t < q^k$ and all $x \in K$,

$$\chi_{rq^k+t}(x) = \chi_{rq^k}(x)\chi_t(x)$$

and

$$\chi_{rq^k}(x) = \chi_r(\beta^{-k}x).$$

Then it follows that

Lemma 2. ([10; p.87]). For $r \in Z^+, k \in Z^+, 0 \le t < q^k$,

$$D_{rq^k+t}(x) = D_{q^k}(x)D_r(\beta^{-k}x) + \chi_r(\beta^{-k}x)D_t(x).$$

Noting that $O/P \cong GF(q), q = p^c$, take $\{1 = \varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{c-1}\} \subset O^* = O\backslash P$ so that $GF(q) \cong span\{\varepsilon_j\}_{j=0}^{c-1}$ is a c-dimensional linear space over $GF(p)$. A straightforward verification gives

Lemma 3. For $k \ge 1$,

$$D_{q^k}(x) = \prod_{s=0}^{k-1} D_q(\beta^{-s}x) = \prod_{s=0}^{k-1}\prod_{j=0}^{c-1} D_p(\varepsilon_j\beta^{-s}x).$$

Now let $n = r_kq^k + r_{k-1}q^{k-1} + \cdots + r_1q + r_0, r_k \ne 0$ and $0 \le r_i < q, i \in \{0, 1, \ldots, k\}$. Then $q^k \le n < q^{k+1}$ and $k \le \log_q n < k+1$. Moreover, we define $n^{(i)}$ so that they satisfy:

$$\begin{aligned}&n = n^{(1)}q + r_0, n^{(1)} = n^{(2)}q + r_1, \cdots,\\ &n^{(k-1)} = n^{(k)}q + r_{k-1} \text{ and } n^{(k)} = r_k.\end{aligned}$$

From the recurrence formula in Lemma 2 , we obtain

Lemma 4. Let $n, n^{(i)}$ be as above . Then

$$\begin{aligned}D_n(x) = \ & D_{q^k}(x)D_{r_k}(\beta^{-k}x) + D_{q^{k-1}}(x)D_{r_{k-1}}(\beta^{-(k-1)}x)\chi_{n^{(k)}}(\beta^{-k}x)+\\ & D_{q^{k-2}}(x)D_{r_{k-2}}(\beta^{-(k-2)}x)\chi_{n^{(k-1)}}(\beta^{-(k-1)}x) + \cdots +\\ & D_q(x)D_{r_1}(\beta^{-1}x)\chi_{n^{(2)}}(\beta^{-2}x)+\\ & D_{r_0}(x)\chi_{n^{(1)}}(\beta^{-1}x).\end{aligned}$$

Choose the base values of χ as follows: if $Char\, K = p$,

$$\chi(\varepsilon_\mu\beta^{-j}) = \begin{cases} \exp(\frac{2\pi i}{p}), & \mu = 0 \text{ and } \ j = 1,\\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \ne 1;\end{cases} \tag{5}$$

if $Char\, K = 0$,

$$\chi(\varepsilon_\mu\beta^{-j}) = \begin{cases} \exp(\frac{2\pi i}{p^j}), & \mu = 0, j \in N,\\ 1, & \mu = 1, \cdots, c-1, j \in N,\end{cases} \tag{6}$$

where $c = [K : K_p]$ denotes the dimension of extension over K_p.

These formulas are from [12]. For the character formulas of the more general a-adic groups we refer to [2] or [4]. Now we state the central lemma of this note as follows.

Lemma 5. Let $k \ge 1$.

(i) If $Char\, K = p$, $D_{q^k}(x) \geq 0$ and

$$\int_{|x|=q} D_{q^k}(x)dx = q - 1.$$

(ii) If $Char\, K = 0$,

$$\int_{|x|=q} |D_{q^k}(x)|dx = A^c k^c,$$

where $A \leq 2p^2$ for $k \geq 1$ and $A \geq (\frac{p-1}{p})^2$ for large $k = k(p)$ depending on p only.

The proof turns out to be of some complexity in the sense of analysis and of some interest in the sense of algebra, especially for the p-adic cases. We shall give the complete proof in the last section to this note.

Theorem 1. (i) If $Char\, K = p$,

$$\int_{|x|=q} |D_n(x)|dx = O(\log n).$$

(ii)If $Char\, K = 0$,

$$\int_{|x|=q} |D_n(x)|dx = O((\log n)^{c+1}).$$

Moreover, the above estimate for D_n is independent of the choice of a character χ' that is trivial on O but nontrivial on P^{-1}.

Proof. Note that any character χ' that is trivial on O but nontrivial on P^{-1} can be written in the form

$$\chi'(x) = \chi_\xi(x)$$

for some $\xi \in O^*$. A priori, the last assertion holds. Let $Char\, K = 0$. By Lemma 4 and Lemma 5 we have for $n = \sum_{s=0}^k r_s q^s, 0 \leq r_s < q$,

$$\begin{aligned} \int_{|x|=q} |D_n(x)|dx &\leq \sum_{s=0}^k r_s \int_{|x|=q} |D_{q^s}(x)|dx \\ &\leq q(2p^2)^c \sum_{s=0}^k (s+1)^c = O(k^{c+1}) = O((\log n)^{c+1}). \end{aligned}$$

This establishes (ii). Similarly (i) follows. □

Let $X(O) = L^p(O), 1 \leq p < \infty$ or $X(O) = C(O)$ for $p = \infty$; $X(K)$ the obvious analog of $X(O)$ on K (in case of no confusion we will write X in place of $X(O)$ or $X(K)$, later on). The *Lipschitz α-classs on X* is defined by

$$Lip(\alpha, X) = \{f \in X : \sup_{h \neq 0} \frac{\|f(\cdot + h) - f(\cdot)\|_X}{|h|^\alpha} < \infty\}.$$

From [7], $Lip(\alpha, L^r(K)) = \Lambda(\alpha, r, \infty)$, a special case of the generalized Lipschitz spaces on K.

Once the estimate for D_n is obtained, we may state and prove the following theorem.

Theorem 2. Let $f \in Lip(\alpha, X(O)), \alpha > 0$. Then

(i) if $0 < \alpha < \nu\delta$,

$$\|R_{\mu,\lambda,n} f - f\|_{X(O)} = O(n^{-\alpha}),$$

(ii) if $\alpha = \nu\delta$,

$$\|R_{\mu,\lambda,n}f - f\|_{X(O)} = O(n^{-\alpha}(\log n)^{Z(K)+1}),$$

(iii) if $\alpha > \nu\delta$,

$$\|R_{\mu,\lambda,n}f - f\|_{X(O)} = O(n^{-\nu\delta}(\log n)^{Z(K)}).$$

Moreover, we have the following saturation estimate

(iv) $\|R_{\mu,\lambda,n}f - f\|_{X(O)} = o(n^{-\mu})$ implies $f =$ constant *a.e.*,

where $\nu = \min(1,\mu), \delta = \min(1,\lambda)$ and

$$Z(K) = \begin{cases} 1 & \text{if } Char\ K = p, \\ c+1 & \text{if } Char\ K = 0. \end{cases}$$

Suppose $q^\ell \le n < q^{\ell+1}, \ell \in N$. For each $k \in \{1, 2, \cdots, \ell\}$, let $n = rq^k + t, 0 \le t < q^k$ so that $q^{\ell-k} \le r < q^{\ell-k+1}$. The proof of Theorem 2 depends on the estimation of the integral of the kernel on certain "shells".

Lemma 6. Let $n, k, r, \ell, \nu, \delta, Z(K)$ be as above. We have

$$\int_{|x|=q^{-k+1}} |\gamma_{\mu,\lambda,n}(x)|dx = O(r^{-\nu\delta}(\log r)^{Z(K)}).$$

Proof. Rewrite $\gamma_{\mu,\lambda,n}$ using Lemma 1:

$$\gamma_{\mu,\lambda,n}(x) = \sum_{\rho=0}^{r-1}\sum_{\tau=0}^{q^k-1}(1-(\frac{\rho q^k+\tau}{n})^\mu)^\lambda \chi_{\rho q^k+\tau}(x) + \sum_{\tau=0}^{t-1}(1-(\frac{rq^k+\tau}{n})^\mu)^\lambda \chi_{rq^k+\tau}(x).$$

Then we have

$$\begin{array}{rl} \int_{|x|=q^{-k+1}} |\gamma_{\mu,\lambda,n}(x)|dx & \le \int_{|x|=q^{-k+1}} |\sum_{\rho=0}^{r-1}\sum_{\tau=0}^{q^k-1}(1-(\frac{\rho q^k+\tau}{n})^\mu)^\lambda \chi_{\rho q^k+\tau}(x)|dx \\ & + \int_{|x|=q^{-k+1}} |\sum_{\tau=0}^{t-1}(1-(\frac{rq^k+\tau}{n})^\mu)^\lambda \chi_{rq^k+\tau}(x)|dx \\ & \equiv I_1 + I_2. \end{array}$$

We estimate I_1, I_2 respectively. Note that if $x \in O$,

$$D_{q^k}(x) = q^k \Phi_k(x), \tag{7}$$

so we have

$$\begin{aligned} I_1 &= \int_{|x|=q^{-k+1}} |\sum_{\rho=0}^{r-1}\sum_{\tau=0}^{q^k-1}[(1-(\frac{\rho q^k+\tau}{n})^\mu)^\lambda - (1-(\frac{\rho q^k}{n})^\mu)^\lambda]\chi_\tau(x)\chi_\rho(\beta^{-k}x)|dx \\ &\le \sum_{\tau=0}^{q^k-1}\int_{|x|=q^{-k+1}} |\sum_{\rho=0}^{r-1}[(1-(\frac{\rho q^k+\tau}{n})^\mu)^\lambda - (1-(\frac{\rho q^k}{n})^\mu)^\lambda]\chi_\rho(\beta^{-k}x)|dx \\ &= q^{-k}\sum_{\tau=0}^{q^k-1}\int_{|x|=q} |\sum_{\rho=0}^{r-1}[(1-(\frac{\rho q^k+\tau}{n})^\mu)^\lambda - (1-(\frac{\rho q^k}{n})^\mu)^\lambda]\chi_\rho(x)|dx \\ &\equiv q^{-k}\sum_{\tau=0}^{q^k-1} A_\tau. \end{aligned} \tag{8}$$

Claim. For each $\tau, 0 \le \tau < q^k$,

$$A_\tau \le c_{\mu,\lambda} r^{-\nu\delta} (\log r)^{Z(K)}, \tag{9}$$

where $c_{\mu,\lambda} > 0$ is independent of τ; $\nu, \delta, Z(K)$ are given as in Lemma 6.

The proof of (9) involves an application of Theorem 1 and the crucial observation that the sequence

$$\begin{array}{ll} b_\rho & = (1-(\frac{(\rho-1)q^k+\tau}{n})^\mu)^\lambda - (1-(\frac{(\rho-1)q^k}{n})^\mu)^\lambda \\ & -(1-(\frac{\rho q^k+\tau}{n})^\mu)^\lambda + (1-(\frac{\rho q^k}{n})^\mu)^\lambda \end{array}$$

changes sign at most once for $1 \le \rho \le r-1$. To see this , let

$$a_{\rho,\tau} = (1-(\frac{\rho q^k+\tau}{n})^\mu)^\lambda - (1-(\frac{\rho q^k}{n})^\mu)^\lambda.$$

By Taylor's formula till second order, there exist some $y_\rho \in (\frac{(\rho-1)q^k}{n}, \frac{\rho q^k+\tau}{n})$ and $z_\rho \in (\frac{q^k-\tau}{n}, \frac{q^k+\tau}{n})$ such that

$$b_\rho = a_{\rho-1,\tau} - a_{\rho,\tau} = \lambda\mu y_\rho^{\mu-2}(1-y_\rho^\mu)^{\lambda-2}(\frac{\tau}{n}z_\rho)[\mu-1+(1-\lambda\mu)y_\rho^\mu].$$

Since the function $f(x) = \mu - 1 + (1-\lambda\mu)x^\mu$ changes sign at most once on $[0,1]$, so does b_ρ :

$$b_{k_0-1}b_{k_0} \le 0$$

for some $k_0 \in \{2, \dots, r-1\}$, say. It follows that

$$\sum_{\rho=1}^{r-1} |b_\rho| = |\sum_{\rho=1}^{k_0-1} b_\rho| + |\sum_{\rho=k_0}^{r-1} b_\rho| \equiv B_1 + B_2, \tag{10}$$

where

$$B_1 = |a_{0,\tau} - a_{k_0-1,\tau}|,\ B_2 = |a_{k_0-1,\tau} - a_{r-1,\tau}|.$$

Now by using the following inequality for $\sigma > 0, 0 \le a, b \le 1$,

$$|a^\sigma - b^\sigma| \le \begin{cases} |a-b|^\sigma & if\ 0 < \sigma < 1, \\ \sigma|a-b| & if\ \sigma \ge 1, \end{cases} \tag{11}$$

we have

$$|a_{0,\tau}| = (1-(\frac{\tau}{n})^\mu)^\lambda - 1 \le c_1(\frac{\tau}{n})^{\mu\delta} \le c_1 r^{-\mu\delta}, \tag{12}$$

and for $1 \le \rho \le r-1$,

$$\begin{aligned} |a_{\rho,\tau}| &= |(1-(\frac{\rho q^k+\tau}{n})^\mu)^\lambda - (1-(\frac{\rho q^k}{n})^\mu)^\lambda| \le c_1[(\frac{\rho q^k+\tau}{n})^\mu - (\frac{\rho q^k}{n})^\mu]^\delta \\ &\le c_1 c_2^\delta (\frac{\tau}{n})^{\nu\delta} \le c_1 c_2^\delta r^{-\nu\delta}, \end{aligned} \tag{13}$$

where we note that $\frac{\tau}{n} \le \frac{q^k}{n} \le \frac{1}{r}$ and $c_1 = \max(1,\lambda),\ c_2 = \max(1,\mu)$.
Combining (10),(12),(13) and Theorem 1, we deduce by summation by parts,

$$\begin{array}{ll} A_\tau & = \int_{|x|=q} |\sum_{\rho=0}^{r-1} a_{\rho,\tau}\chi_\rho(x)|dx \le \sum_{\rho=1}^{r-1} |b_\rho| \int_{|x|=q} |D_\rho(x)|dx + |a_{r-1,\tau}| \int_{|x|=q} |D_r(x)|dx \\ & \le (B_1 + B_2 + |a_{r-1,\tau}|)O((\log r)^{Z(K)}) = O(r^{-\nu\delta}(\log r)^{Z(K)}). \end{array}$$

This proves (9).

To estimate I_2 , we deduce by (11)

$$\begin{aligned} I_2 &= \int_{|x|=q^{-k+1}} |\sum_{\tau=0}^{t-1}(1-(\frac{rq^k+\tau}{n})^{\mu})^{\lambda}\chi_{rq^k+\tau}(x)|dx \\ &\leq q^{-k+1}[\sum_{\tau=0}^{t-1}(1-(\frac{rq^k+\tau}{n})^{\mu})^{\lambda}] \leq c_2^{\lambda}q^{-k+1}\sum_{\tau=0}^{t-1}(\frac{t-\tau}{n})^{\nu\lambda} \\ &= O(r^{-\nu\lambda}). \end{aligned} \tag{14}$$

Combining (8),(9) and (14) gives

$$\int_{|x|=q^{-k+1}} |\gamma_{\mu,\lambda,n}(x)|dx = O(r^{-\nu\delta}(\log r)^{Z(K)}).$$

This completes the proof of Lemma 6. $\square$

Outline of the proof of Theorem 2. As in the classical case [14], the proof reduces to an estimation of the $\alpha-$order moment of $\gamma_{\mu,\lambda,n}$. But this follows readily from Lemma 6. Using the same notations as in Lemma 6, we have

$$\begin{aligned} \int_{|x|=q^{-k+1}} |x|^{\alpha}|\gamma_{\mu,\lambda,n}(x)|dx &= q^{(-k+1)\alpha}\int_{|x|=q^{-k+1}} |\gamma_{\mu,\lambda,n}(x)|dx \\ &= q^{-k\alpha}O(r^{-\nu\delta}(\log r)^{Z(K)}) = n^{-\alpha}O(r^{\alpha-\nu\delta}(\log r)^{Z(K)}). \end{aligned}$$

Therefore,

$$\begin{aligned} m_{\alpha} &\equiv \textstyle\int_O |x|^{\alpha}|\gamma_{\mu,\lambda,n}(x)|dx = \int_{|x|\leq q^{-\ell}} + \int_{q^{-\ell}<|x|\leq 1} \\ &\leq n\textstyle\int_{|x|\leq q^{-\ell}} |x|^{\alpha}dx + \sum_{k=1}^{\ell}\int_{|x|=q^{-k+1}} |x|^{\alpha}|\gamma_{\mu,\lambda,n}(x)|dx \\ &\leq nq^{-\ell(\alpha+1)} + n^{-\alpha}\textstyle\sum_{k=1}^{\ell} O(r^{\alpha-\nu\delta}(\log r)^{Z(K)}) \\ &= O(n^{-\alpha}) + n^{-\alpha}\textstyle\sum_{j=0}^{\ell-1} O(q^{j(\alpha-\nu\delta)}(j+1)^{Z(K)}) \\ &= \begin{cases} O(n^{-\alpha}) & 0<\alpha<\nu\delta, \\ O(n^{-\alpha})(\log n)^{Z(K)+1} & \alpha=\nu\delta, \\ O(n^{-\nu\delta}(\log n)^{Z(K)}) & \alpha>\nu\delta. \end{cases} \end{aligned}$$

$\square$

Remark. With D_n behaving badly on $P^{-1}\backslash O$, the approximation degree by Riesz means (of critical and higher order) is lower than in the trigonometric case. Let us give a comparison with Leindler's result [6; Theorem E] for $X = C(O)$. Fix $\lambda = 1, \mu > 0$ and $0 < \alpha \leq 1$ first, then

(i) if $\alpha < \mu$, we have the same approximation degree $O(n^{-\alpha})$,

(ii) if $\alpha = \mu$, Leindler's degree is $O(n^{-\alpha}\log n)$ while ours is $O(n^{-\alpha}(\log n)^{Z(K)+1})$.

In case of $\alpha > 1$, set $\alpha' = \alpha - \beta$, where $\beta \in N$ is taken so that $\alpha' \in (0, 1]$. From Theorem 4 in the next section, we see the following relation holds:

$$f \in Lip(\alpha, X) \Longleftrightarrow D_X^{\beta}f \in Lip(\alpha', X).$$

Unfortunately, this fact will not provide the comparable result as in [6 ;Theorem E] for β-order differentiable functions, since the degree of approximation we get for $R_{\mu,\lambda,n}f$ is at most $O(n^{-\nu\delta}) \geq O(n^{-1})$.

3. RIESZ KERNELS OF TYPE I

One may expect that higher degrees of approximation can be obtained with some more properly constructed "Riesz type kernels." In effect, there are such ones . If $\lambda = 1$ and k, n are replaced by $u(k), u(n)$ in (2), we have a new type of kernels defined as in (4), which will be called *Riesz kernels of order μ and of type I.* Correspondingly, the convolutions $\tilde{R}_{\mu,n}f(x) = \tilde{\gamma}_{\mu,n} * f(x)$ are called *Riesz means of order μ and of type I.* Similar names apply to other kernels or means (see Definition 3 at the end of this section).

Estimation for $\tilde{\gamma}_{\mu,n}$ becomes very simple by applying the method used in proving Lemma 6. Substituting $\tilde{\gamma}_{\mu,n}(x)$ for $\gamma_{\mu,1,n}(x)$ in the proof of Lemma 6 and noting that if $1 \leq \rho \leq r, 0 \leq \tau < q^k, |u(\rho q^k + \tau)| = |u(\rho q^k)|$, one finds $I_2 = 0$ and

$$\begin{aligned} I_1 &= \textstyle\int_{|x|=q^{-k+1}} |\sum_{\tau=0}^{q^k-1} |u(\tau)u(n)^{-1}|^{\mu} \chi_\tau(x)| dx \\ &\leq q|u(q^k)u(n)^{-1}|^{\mu} = O(r^{-\mu}), \end{aligned}$$

and so

$$\int_{|x|=q^{-k+1}} |\tilde{\gamma}_{\mu,n}(x)| dx = O(r^{-\mu}).$$

Therefore the next theorem follows for Riesz means of type I.

Theorem 3. Let $f \in Lip(\alpha, X), \alpha > 0, X = X(O)$, then

(i) if $\alpha < \mu$, $\|\tilde{R}_{\mu,n}f - f\|_X = O(n^{-\alpha})$,

(ii) if $\alpha = \mu$, $\|\tilde{R}_{\mu,n}f - f\|_X = O(n^{-\alpha}\log n)$,

(iii) if $\alpha > \mu$, $\|\tilde{R}_{\mu,n}f - f\|_X = O(n^{-\mu})$,

(iv) $\|\tilde{R}_{\mu,n}f - f\|_X = o(n^{-\mu})$ implies $f = const.\ a.e.$

The remaining part of this section is devoted to giving a heuristic comparison with Leindler's result for higher order Lipschitz functions in terms of $\tilde{R}_{\mu,n}f$. To this end we need the following definition of strong derivative of order α for functions in $X(O)$, which may be viewed as the restriction of its counterpart on K defined by Onneweer [7]. Other equivalent definitions (in the strong sense) of fractional derivatives appeared in [13], [4] in some different forms (cf.[13;Theorem 3] and [4;Theorem 3.6]). We choose Onneweer's definition on O because it makes some computations easier.

Definition 1. Let $\alpha > 0$ and $f \in L_{loc}(O)$. Let

$$\tilde{E}_{N,\alpha}f(x) = \sum_{\ell=0}^{N-1} (q^{(\ell+1)\alpha} - q^{\ell\alpha})(f - f(\cdot, -\ell))(x), \tag{15}$$

where $f(\cdot, k) = f * R(\cdot, k), R(\cdot, k) = q^{-k}\Phi_{-k}$. We say $f \in X(O)$ has a strong derivative in $X(O)$ of order α, if $D^\alpha_{X(O)}f := \lim \tilde{E}_{N,\alpha}f$ exists in $X(O)$. The space $\mathcal{D}\ (D^\alpha_X) = \{f \in X : D^\alpha_X f$ exists $\}$, where $X = X(O)$.

Definition 2[7]. Let

$$E_{N,\alpha}f(x) = \sum_{\ell=-\infty}^{N-1} (q^{(\ell+1)\alpha} - q^{\ell\alpha})(f - f(\cdot, -\ell))(x).$$

We say $f \in X(K)$ has a strong derivative in $X(K)$ of order α, if $D^{\alpha}_{X(K)}f =: \lim_{N\to\infty} E_{N,\alpha}f$ exists in $X(K)$. The space $\mathcal{D}(D^{\alpha}_{X(K)}) = \{f \in X(K) : D^{\alpha}_{X(K)}f \text{ exists }\}$.

Lemma 7. If $f \in X(O)$, define $F(x) = f(x)\Phi_0$. Then $D^{\alpha}_{X(K)}F$ exists in $X(K) \Longleftrightarrow D^{\alpha}_{X(O)}f$ exists in $X(O)$.

The lemma easily follows from the following observation: Let F, f be as in the lemma, a simple computation by Definition 1 and Definition 2 gives

$$\begin{aligned} E_{N,\alpha}F(x) &= \tilde{E}_{N,\alpha}f(x) + \textstyle\int_O f(t)dt\{-\Phi_0(x) + \frac{1-q^{-1}}{1-q^{-(1+\alpha)}}\Phi_0(x) \\ &+\frac{1-q^{\alpha}}{1-q^{-(1+\alpha)}}\frac{1-\Phi_0(x)}{|x|^{1+\alpha}}\} + f(x)\Phi_0(x) \\ &= \tilde{E}_{N,\alpha}f(x) - \textstyle\int_O f(t)dt[(1-|\cdot|^{\alpha})\Phi_0](x) + f(x)\Phi_0, \end{aligned}$$

where we note that $\tilde{E}_{N,\alpha}f(x)$ is supported in O and the function $(1-\Phi_0)|x|^{-1-\alpha}$ is in $Lip(\beta, L^r(K)) \cap Lip(\beta, C(K))$ $(\forall \beta > 0, 1 \le r < \infty)$ and hence in $\mathcal{D}(D^{\infty}_{X(K)}) \equiv \cap_{\alpha>0}\ \mathcal{D}(D^{\alpha}_{X(K)})$ from the first inclusion in (16), below. Therefore Lemma 7 holds. □

In [4;Theorem 3.5], it is proved that if $0 < \alpha < \beta$,

$$Lip(\beta, X) \subset \mathcal{D}(D^{\alpha}_X) \subset Lip(\alpha, X), \tag{16}$$

for $X = X(K)$ (although the definiton of derivative was given in a different form, the same argument in proving (16) applies to Onneweer's derivative as defined in Definition 2). To be precise, we remark that the second inclusion in (16) is only valid for $X = L^r(K), 1 \le r < \infty$ and a standard proof for the case $2 < r < \infty$ should follow via the next theorem e.g. in [5; Theorem 2.9]:

Theorem O. Let $\alpha > 0$ and $f \in L^r(K)$ with $2 < r < \infty$. Then $D^{\alpha}_{L^r}f$ exists if and only if there exists a $g \in L^r(K)$ so that for all $\phi \in \mathcal{D}(D^{\alpha}_{r'})$, where $1/r + 1/r' = 1$, we have

$$\int_K f(x)D^{\alpha}_{r'}\phi(x)dx = \int_K g(x)\phi(x)dx,$$

where $D^{\alpha}_{L^r}f = g$ is also called the *weak derivative of f of order α.* In other words, the strong derivative and weak derivative of f of order α coincide.

So with a slight modification of the proof of Theorem 3.5 in [4], one easily has

Theorem 4. Let $X = L^r(K), 1 \le r < \infty$. If $0 < \beta < \alpha$, then

$$f \in Lip(\alpha, X) \Longleftrightarrow D^{\beta}_X f \textit{ exists and } D^{\beta}_X f \in Lip(\alpha - \beta, X).$$

In addition, if $X = C(K)$, the one way implication "⇒" as above holds .

We sketch the proof here. The "⇒" part follows quite similarly from the argument given in the proof of the first inclusion in Theorem 3.5[4]; hence it is omitted.

Let $V_{N,\alpha} = \sum_{\ell=-N+1}^{\infty} q^{-\ell\alpha}(R(\cdot,-\ell) - R(\cdot,-\ell+1))(x)$. Then $\|V_{N,\alpha}\|_{L^1(K)} = O(q^{N\alpha})$ and $(V_{N,\alpha})(x) = |x|^{-\alpha}(1-\Phi_N)$. Proof of the "⇐" part is based on the following identity for proving the second inclusion in Theorem 3.5[4]: if $|h| = q^{-N}$,

$$f(x-h) - f(x) = V_{-N,\alpha} * (D^{\alpha}f(x-h) - D^{\alpha}f(x)), \tag{17}$$

which holds for $X = L^r(K), 1 \le r < \infty$, in virtue of Theorem O above . This ends the proof. □

Noting that from the discussion following Lemma 7 we may conclude

$$\mathcal{D}(\ D^{\alpha}_{L^r(K),comp}) \cup \mathcal{D}(D^{\alpha}_{C(K),comp}) \subset \mathcal{D}(D^{\alpha}_{L^1(K)}),$$

where the space $\mathcal{D}(D^{\alpha}_{X(K),comp})$ consists of functions in $\mathcal{D}(D^{\alpha}_{X(K)})$ with compact support, we see (17) is also valid for functions in $\mathcal{D}(D^{\alpha}_{C(K),comp})$. Thus we obtain the analog of Theorem 4 on O.

Theorem 4'. Let $X = L^r(O), 1 \leq r < \infty$ or $X = C(O)$. If $0 < \beta < \alpha$, then

$$f \in Lip(\alpha, X) \Longleftrightarrow D^{\beta}_{X} f \ \textit{exists and}\ D^{\beta}_{X} f \in Lip(\alpha - \beta, X).$$

An equivalent version of Theorem 3 follows from Theorem 4'.

Theorem 3'. Let $D^{\beta}_{X} f \in Lip(\alpha, X), \alpha > 0, \beta > 0, X = X(O)$. Then

(i) $\|\tilde{R}_{\mu,n} f - f\|_X = O(n^{-\beta-\alpha})$ if $\beta + \alpha < \mu$,

(ii) $\|\tilde{R}_{\mu,n} f - f\|_X = O(n^{-\beta-\alpha}\log n)$ if $\beta + \alpha = \mu$,

(iii) $\|\tilde{R}_{\mu,n} f - f\|_X = O(n^{-\mu})$ if $\beta + \alpha > \mu$,

Remark. The estimate in (ii) is sharp at least for $X = C(O)$. One way to see this is to note that by the Poisson summation formula on K [10;p.92] $\tilde{\gamma}_{\mu,n}(q^{\ell} \leq n < q^{\ell+1})$ is essentially the restriction of the function $h_{\ell} = |\beta^{-\ell}| h(\beta^{-\ell} x)$ on O, where $h(x) = [(1 - |\cdot|^{\alpha})\Phi_0](x) \in L^1(K)$. Let $f = |x|^{\alpha}\Phi_0 \in Lip(\alpha, C)$, i.e., $D^{\beta}_{C} f \in Lip(\alpha - \beta)$. It is not difficult to show that there exists a constant $c_{\alpha} > 0$ such that for large n,

$$|h_{\ell} * f(0)| \geq c_{\alpha}(n^{-\alpha}\log n).$$

This will imply $\|\tilde{R}_{\alpha,n} f - f\|_C \geq c_{\alpha}(n^{-\alpha}\log n)$ as well. $\square$

We end this section with the definitions of Cesáro kernels and Abel-Poisson kernels of type I, corresponding to the trigonometric case.

Definition 3. The *Cesáro kernels of order λ and of type I* are the functions

$$\tilde{K}_{\lambda,n}(x) = (A^{\lambda}_{|u(n)|})^{-1} \sum_{k=0}^{n} A^{\lambda}_{|u(n-k)|}\chi_k(x),$$

where the Cesáro numbers ([14;p.77])

$$A^{\lambda}_{n} = \binom{n+\lambda}{n} = \frac{n^{\lambda}}{\Gamma(\lambda+1)}(1 + O(\frac{1}{n})).$$

The *Abel-Poisson kernels of type I* are the functions

$$\tilde{P}(r, x) = \sum_{k=0}^{\infty} r^{|u(k)|}\chi_k(x),$$

where $\tilde{P}(r, x)$ are defined on $[0, 1) \times O$.

One reason why we define "kernels of type I" as above is that by our approach these kernels are easier to estimate and their associated means yield some satisfying approximation properties, namely, the rates of convergence for Lipschitz functions are the same as in the classical case [14]. The interested reader can deduce these properties without difficulty.

4. PROOF OF LEMMA 5

In this section, in order to prove Lemma 5, we develop a unified approach to inspect the behavior of $D_{q^k}(x)$ on $P^{-1}\backslash O$ for the p-series and p-adic cases, the latter being of more significance . Write $K_q = K$ as in [12], $q = p^c$. Recall that $\{1 = \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{c-1}\} \subset O^*$ is chosen so that $GF(p^c) \cong span\{\varepsilon_j\}_{j=0}^{c-1}$ over $GF(p)$. Since $O/P \cong GF(p^c), GF(p^c)$ will be identified with a subset of O if $Char\, K_q = p$ or a subset of $O(mod\ P)$ if $Char\, K_q = 0$.

I. K_q is a finite algebraic extension of a p-series field ($Char\, K_q = p$).

Since $D_{q^k}(x)$ is constant on cosets of P^k in P^{-1}, we let

$$x \in \alpha_{-1}\beta^{-1} + \alpha_0\beta^0 + \cdots + \alpha_{k-1}\beta^{k-1} + P^k,$$

where $\alpha_i = \alpha_{i,0}\varepsilon_0 + \alpha_{i,1}\varepsilon_1 + \cdots + \alpha_{i,c-1}\varepsilon_{c-1}, \alpha_{i,\ell} \in GF(p) \cong Z_p = \{0, 1, \ldots, p-1\}$. To estimate $D_{q^k}(x)$ we shall compute

$$D_q(\beta^{-s}x),\ s = 0, 1, \ldots, k-1.$$

In view of (5),

$$\chi_{\varepsilon_j}(\beta^{-1}x) = \chi_{\varepsilon_j}(\alpha_{-1}\beta^{-2} + \alpha_0\beta^{-1}) = \chi(\alpha_0\varepsilon_j\beta^{-1}) = 1 - \exp\frac{2\pi i}{p}\gamma_{0,0}^j,$$

where $\alpha_0\varepsilon_j = \gamma_{0,0}^j\varepsilon_0 + \gamma_{0,1}^j\varepsilon_1 + \cdots + \gamma_{0,c-1}^j\varepsilon_{c-1}$ for some unique $\gamma_{0,\ell}^j \in GF(p), j, \ell \in \{0, 1, \ldots, c-1\}$. We have by Lemma 3

$$\begin{aligned} D_q(x) &= \prod_{j=0}^{c-1} D_p(\varepsilon_j x) = \prod_{j=0}^{c-1} \frac{1 - \chi_{\varepsilon_j}^p(\beta^{-1}x)}{1 - \chi_{\varepsilon_j}(\beta^{-1}x)} \\ &= \prod_{j=0}^{c-1} \frac{1 - \exp 2\pi i\gamma_{0,0}^j}{1 - \exp\dfrac{2\pi i}{p}\gamma_{0,0}^j}. \end{aligned}$$

From this we see

$$D_q(x) \neq 0 \Leftrightarrow \gamma_{0,0}^j = 0\ for\ all\ j \Leftrightarrow \alpha_0 = 0. \tag{18}$$

The last equivalence we prove as follows. Suppose $\gamma_{0,0}^j = 0$ for all j. Since $span\{\varepsilon_j\}_{j=0}^{c-1} = GF(p^c)$ over $GF(p)$, if $\alpha_0 \neq 0$, then we would have $span\{\alpha_0\varepsilon_j\}_{j=0}^{c-1} = GF(p^c)$ over $GF(p)$. But this leads to the contradiction that $span\{\alpha_0\varepsilon_j\}_{j=0}^{c-1} = span\{\gamma_{0,1}^j\varepsilon_1 + \cdots + \gamma_{0,c-1}^j\varepsilon_{c-1}\}_{j=0}^{c-1}$ $= span\{\varepsilon_j\}_{j=1}^{c-1} \subseteq GF(p^c)$. So we must have $\alpha_0 = 0$ and (18) follows (the other direction being obvious).

Hence

$$D_q(x) = \begin{cases} p^c & \text{if } x \in \alpha_{-1}\beta^{-1} + P, \alpha_{-1} \in GF(p^c), \\ 0 & \text{if otherwise in } P^{-1}. \end{cases}$$

Next we compute $D_q(\beta^{-1}x)$. From (5) we have for each j,

$$\chi_{\varepsilon_j}(\beta^{-2}x) = \chi_{\varepsilon_j}(\alpha_{-1}\beta^{-3} + \alpha_0\beta^{-2} + \alpha_1\beta^{-1}) = \chi(\alpha_1\varepsilon_j\beta^{-1}) = \exp\frac{2\pi i}{p}\gamma_{1,0}^j,$$

where $\alpha_1\varepsilon_j = \gamma^j_{1,0}\varepsilon_0 + \gamma^j_{1,1}\varepsilon_1 + \cdots + \gamma^j_{1,c-1}\varepsilon_{c-1} \in GF(p^c)$ for some unique $\gamma^j_{1,\ell} \in GF(p)$. Thus

$$\begin{aligned} D_q(\beta^{-1}x) &= \prod_{j=0}^{c-1} D_p(\varepsilon_j\beta^{-1}x) = \prod_{j=0}^{c-1} \frac{1-\chi^p_{\varepsilon_j}(\beta^{-2}x)}{1-\chi_{\varepsilon_j}(\beta^{-2}x)} \\ &= \prod_{j=0}^{c-1} \frac{1-\exp 2\pi i\gamma^j_{1,0}}{1-\exp\dfrac{2\pi i}{p}\gamma^j_{1,0}}. \end{aligned}$$

It follows that (similar to the argument below (18)),

$$D_q(\beta^{-1}x) \neq 0 \Leftrightarrow \gamma^j_{1,0} = 0 \ for\ all\ j \Leftrightarrow \alpha_1 = 0.$$

By Lemma 3 we have

$$D_{q^2}(x) = D_q(x)D_q(\beta^{-1}x) \neq 0 \Leftrightarrow \alpha_0 = \alpha_1 = 0,$$

and so,

$$D_{q^2}(x) = \begin{cases} (p^c)^2 & \text{if } x \in \alpha_{-1}\beta^{-1} + P^2, \\ 0 & \text{if otherwise in } P^{-1}. \end{cases}$$

A consecutive process of the above argument yields

$$D_{q^k}(x) = \prod_{s=0}^{k-1} D_q(\beta^{-s}x) = \begin{cases} (p^c)^k & \text{if } x \in \alpha_{-1}\beta^{-1} + P^k, \\ 0 & \text{if otherwise in } P^{-1}, \end{cases}$$

where we observe that

a)$D_{q^k}(x) \geq 0$ if $x \in P^{-1}$,

b) there are $(p^c - 1)$ tuples $(\alpha_{-1,0}, \ldots, \alpha_{-1,c-1})$ such that

$$\alpha_{-1} = \alpha_{-1,0}\varepsilon_0 + \cdots + \alpha_{-1,c-1}\varepsilon_{c-1} \neq 0,\ \alpha_{-1,j} \in GF(p).$$

Thus

$$\begin{aligned} \int_{|x|=q} |D_{q^k}(x)|dx &= \sum_{\substack{(\alpha_{-1},\alpha_0,\ldots,\alpha_{k-1}) \\ \alpha_{-1}\neq 0}} \int_{\alpha_{-1}\beta^{-1}+\cdots+\alpha_{k-1}\beta^{-1}+P^k} D_{q^k}(x)dx \\ &= \sum_{\alpha_{-1}\neq 0} \int_{\alpha_{-1}\beta^{-1}+P^k} q^k dx = q^k \cdot q^{-k}(p^c - 1) = q - 1. \end{aligned}$$

This proves Lemma 5 (i).

II. K_q is a finite algebraic extension of a p-adic field ($Char\, K_q = 0$).

Recall that for $x \in K_q$([12],[10]),

$$x = \sum_{\ell=s}^{\infty} x_\ell \beta^\ell,\ \ s \in Z,$$

where $x_\ell = x_{\ell,0}\varepsilon_0 + x_{\ell,1}\varepsilon_1 + \cdots + x_{\ell,c-1}\varepsilon_{c-1}, x_{\ell,j} \in GF(p)$. The addition and multiplication on K_q are done formally, carrying from left to right.

Similar to Case I, let $x \in \alpha_{-1}\beta^{-1} + \alpha_0\beta^0 + \cdots + \alpha_{k-1}\beta^{k-1} + P^k$, where $\alpha_i = \alpha_{i,0}\varepsilon_0 + \alpha_{i,1}\varepsilon_1 + \cdots + \alpha_{i,c-1}\varepsilon_{c-1}, i \in \{-1, 0, \ldots, k-1\}$. Simple induction shows that for each j, there exist unique $\gamma^j_{-1}, \gamma^j_0, \ldots, \gamma^j_{k-1} \in GF(p^c)$ satisfying the following equations over K_q:

$$\begin{cases} (\alpha_{-1}\beta^{-2} + \alpha_0\beta^{-1})\varepsilon_j &= \gamma^j_{-1}\beta^{-2} + \gamma^j_0\beta^{-1}(mod\ O), \\ (\alpha_{-1}\beta^{-3} + \alpha_0\beta^{-2} + \alpha_1\beta^{-1})\varepsilon_j &= \gamma^j_{-1}\beta^{-3} + \gamma^j_0\beta^{-2} + \gamma^j_1\beta^{-1}(mod\ O), \\ \quad\vdots & \\ (\alpha_{-1}\beta^{-k-1} + \alpha_0\beta^{-k} + \cdots\alpha_{k-1}\beta^{-1})\varepsilon_j &= \gamma^j_{-1}\beta^{-k-1} + \gamma^j_0\beta^{-k} + \cdots\gamma^j_{k-1}\beta^{-1}(mod\ O), \end{cases} \tag{19}$$

where $\gamma_i^j = \gamma_{i,0}^j \varepsilon_0 + \cdots + \gamma_{i,c-1}^j \varepsilon_{c-1}$, $\gamma_{i,\ell}^j \in GF(p)$.

Now we compute $D_p(\varepsilon_j \beta^{-s} x), s = 0, 1, \ldots, k-1, j = 0, 1, \ldots, c-1$. From (6) and the equations (19) we see that: if $x = \alpha_{-1}\beta^{-1} + \alpha_0\beta^0 + \cdots + \alpha_{k-1}\beta^{k-1} (mod\ P^k)$, then

$$\begin{aligned} \chi_{\varepsilon_j}(\beta^{-s-1}x) &= \chi_{\varepsilon_j}(\alpha_{-1}\beta^{-s-2} + \alpha_0\beta^{-s-1} + \cdots + \alpha_s\beta^{-1}) \\ &= \chi(\gamma_{-1,0}^j\beta^{-s-2} + \gamma_{0,0}^j\beta^{-s-1} + \cdots + \gamma_{s,0}^j\beta^{-1}) \\ &= \exp \tfrac{2\pi i}{p^{s+2}}(\gamma_{-1,0}^j + p\gamma_{0,0}^j + \cdots + p^{s+1}\gamma_{s,0}^j); \end{aligned}$$

and so,

$$\begin{aligned} D_p(\varepsilon_j \beta^{-s} x) &= \textstyle\sum_{k=0}^{p-1} (\chi_{\beta^{-1}}(\varepsilon_j \beta^{-s} x))^k = \dfrac{1 - \chi_{\varepsilon_j}^p(\beta^{-s-1}x)}{1 - \chi_{\varepsilon_j}(\beta^{-s-1}x)} \\ &= \dfrac{1 - \exp \dfrac{2\pi i}{p^{s+1}}(\gamma_{-1,0}^j + p\gamma_{0,0}^j + \cdots + p^s\gamma_{s-1,0}^j)}{1 - \exp \dfrac{2\pi i}{p^{s+2}}(\gamma_{-1,0}^j + p\gamma_{0,0}^j + \cdots + p^{s+1}\gamma_{s,0}^j)}. \end{aligned}$$

Observe that if $\gamma_{-1,0}^j \neq 0$,

$$\begin{aligned} \textstyle\prod_{s=0}^{k-1} |D_p(\varepsilon_j \beta^{-s} x)| &= \dfrac{|1 - \exp \dfrac{2\pi i}{p}\gamma_{-1,0}^j|}{|1 - \exp \dfrac{2\pi i}{p^{k+1}}(\gamma_{-1,0}^j + p\gamma_{0,0}^j + \cdots + p^k\gamma_{k-1,0}^j)|} \\ &= \dfrac{\sin \dfrac{\pi}{p}\gamma_{-1,0}^j}{\sin \dfrac{\pi}{p^{k+1}}(\gamma_{-1,0}^j + p\gamma_{0,0}^j + \cdots + p^k\gamma_{k-1,0}^j).} \end{aligned}$$

Hence by Lemma 3,

$$|D_{q^k}(x)| = \prod_{s=0}^{k-1}\prod_{j=0}^{c-1} |D_p(\varepsilon_j \beta^{-s} x)| = \prod_{j=0}^{c-1} H_{\Gamma^j}, \tag{20}$$

where if $\Gamma^j \equiv (\gamma_{-1,0}^j, \gamma_{0,0}^j, \ldots, \gamma_{k-1,0}^j)$,

$$\begin{aligned} H_{\Gamma^j} &= \textstyle\prod_{s=0}^{k-1} |D_p(\varepsilon_j \beta^{-s} x)| \\ &= \begin{cases} \dfrac{\sin \dfrac{\pi}{p}\gamma_{-1,0}^j}{\sin \dfrac{\pi}{p^{k+1}}(\gamma_{-1,0}^j + p\gamma_{0,0}^j + \cdots + p^k\gamma_{k-1,0}^j)} & \text{if } \gamma_{-1,0}^j \neq 0, \\ 0 & \text{if } \gamma_{-1,0}^j = 0 \text{ but } \gamma_{i,0}^j \neq 0 \text{ for some } 0 \leq i \leq k-1, \\ p^k & \text{if } \gamma_{-1,0}^j = \gamma_{0,0}^j = \cdots = \gamma_{k-1,0}^j = 0. \end{cases} \end{aligned}$$

To estimate H_{Γ^j}, note that for each j the $(k+1)$-tuple Γ^j belongs to one of the following cases, where the next inequalities may be applied:

$$1 \leq \frac{x}{\sin x} \leq \frac{\pi}{2} \ if\ x \in (0, \frac{\pi}{2}]\ and$$

$$\sin \frac{\pi}{p} \leq \sin \frac{\pi}{p}\gamma \ \ if\ \gamma \in \{1, \ldots, p-1\}.$$

$Case\,(A)$ If $\gamma^j_{-1,0} \neq 0$ and $0 < \gamma^j_{k-1,0} < p-1$ (when $p=2$, this case vanishes), there are $(p-1)\underbrace{p\cdots p}_{\text{k-1 times}}(p-2)$ tuples in Γ^j such that

$$1 \geq \sin\frac{\pi}{p^{k+1}}(\gamma^j_{-1,0} + p\gamma^j_{0,0} + \cdots + p^k\gamma^j_{k-1,0}) \geq \min(\sin\frac{p-1}{p}\pi, \sin\frac{\gamma^j_{k-1,0}}{p}\pi) = \sin\frac{\pi}{p};$$

and so

$$\frac{2}{p} \leq \sin\frac{\pi}{p} \leq H_{\Gamma^j} \leq \frac{1}{\sin\frac{\pi}{p}} \leq \frac{p}{2}.$$

$Case\,(B_1)$ If $\gamma^j_{-1,0} \neq 0, \gamma^j_{k-1,0} = 0$ but $\gamma^j_{k-2,0} \neq 0$, there are $(p-1)\underbrace{p\cdots p}_{\text{k-2 times}}(p-1)$ tuples in Γ^j such that

$$\sin\frac{\pi}{p} \geq \sin\frac{\pi}{p^{k+1}}(\gamma^j_{-1,0} + p\gamma^j_{0,0} + \cdots + p^{k-1}\gamma^j_{k-2,0}) \geq \sin\frac{\pi}{p^2};$$

and so

$$1 \leq H_{\Gamma^j} \leq \frac{p^2}{2}.$$

$Case\,(B_2)$ If $\gamma^j_{-1,0} \neq 0, \gamma^j_{k-1,0} = \gamma^j_{k-2,0} = 0$ but $\gamma^j_{k-3,0} \neq 0$, there are $(p-1)\underbrace{p\cdots p}_{\text{k-3 times}}(p-1)$ tuples in Γ^j such that

$$\sin\frac{\pi}{p^2} \geq \sin\frac{\pi}{p^{k+1}}(\gamma^j_{-1,0} + p\gamma^j_{0,0} + \cdots + p^{k-2}\gamma^j_{k-3,0}) \geq \sin\frac{\pi}{p^3};$$

and

$$\frac{2}{\pi}p \leq H_{\Gamma^j} \leq \frac{p^3}{2}.$$

$\cdots\cdots$

$Case\,(B_k)$ If $\gamma^j_{-1,0} \neq 0, \gamma^j_{k-1,0} = \gamma^j_{k-2,0} = \cdots = \gamma^j_{0,0} = 0$, there are $(p-1)$ tuples in Γ^j such that

$$\sin\frac{\pi}{p^k} \geq \sin\frac{\pi}{p^{k+1}}\gamma^j_{-1,0} \geq \sin\frac{\pi}{p^{k+1}};$$

and

$$\frac{2}{\pi}p^{k-1} \leq H_{\Gamma^j} \leq \frac{p^{k+1}}{2}.$$

$Case\,(C_1)$ If $\gamma^j_{-1,0} \neq 0, \gamma^j_{k-1,0} = p-1$ and $\gamma^j_{k-2,0} < p-1$, there are $(p-1)\underbrace{p\cdots p}_{\text{k-2 times}}(p-1)$ tuples in Γ^j such that

$$\sin\frac{\pi}{p} = \sin\frac{p-1}{p}\pi \geq \sin\frac{\pi}{p^{k+1}}(\gamma^j_{-1,0} + \cdots + p^{k-1}\gamma^j_{k-2,0} + p^k(p-1))$$

$$\geq \sin(\frac{p-1}{p} + \frac{p-1}{p^2})\pi = \sin\frac{\pi}{p^2};$$

$$1 \leq H_{\Gamma^j} \leq \frac{p^2}{2}.$$

Case (C_2) If $\gamma^j_{-1,0} \neq 0, \gamma^j_{k-1,0} = \gamma^j_{k-2,0} = p-1$ and $\gamma^j_{k-3,0} < p-1$, there are $(p-1)\underbrace{p\cdots p}_{\text{k-3 times}}(p-1)$ tuples in Γ^j such that

$$\sin\frac{\pi}{p^2} = \sin(\frac{p-1}{p} + \frac{p-1}{p^2})\pi$$

$$\geq \sin\frac{\pi}{p^{k+1}}(\gamma^j_{-1,0} + p\gamma^j_{0,0} + \cdots + p^{k-2}\gamma^j_{k-3,0} + p^{k-1}(p-1) + p^k(p-1))$$

$$\geq \sin(\frac{p-1}{p} + \frac{p-1}{p^2} + \frac{p-1}{p^3})\pi = \sin\frac{\pi}{p^3};$$

and

$$\frac{2}{\pi}p \leq H_{\Gamma^j} \leq \frac{p^3}{2}.$$

$\cdots\cdots$

Case (C_k) If $\gamma^j_{-1,0} \neq 0$ and $\gamma^j_{k-1,0} = \gamma^j_{k-2,0} = \cdots = \gamma^j_{0,0} = p-1$, there are $(p-1)$ tuples in Γ^j such that

$$\sin\frac{\pi}{p^k} = \sin(\frac{p-1}{p} + \frac{p-1}{p^2} + \cdots + \frac{p-1}{p^k})\pi$$

$$\geq \sin\frac{\pi}{p^{k+1}}(\gamma^j_{-1,0} + p(p-1)\cdots + p^k(p-1))$$

$$\geq \sin\frac{\pi}{p^{k+1}}((p-1) + p(p-1) + \cdots + p^k(p-1)) = \sin\frac{\pi}{p^{k+1}};$$

and so

$$\frac{2}{\pi}p^{k-1} \leq H_{\Gamma^j} \leq \frac{p^{k+1}}{2}.$$

Case (D) If $\gamma^j_{-1,0} = \gamma^j_{0,0} = \cdots = \gamma^j_{k-1,0} = 0$, there is only one tuple in Γ^j such that

$$H_{\Gamma^j} = p^k.$$

Case (E) If $\gamma^j_{-1,0} = 0$ but $\gamma^j_{i,0} \neq 0$ for some $i \in \{0, 1, \ldots, k-1\}$, then

$$H_{\Gamma^j} = 0.$$

We have

$$\begin{aligned}\int_{|x|\leq q} |D_{q^k}(x)|dx &= \sum_{(\alpha_{-1},\alpha_0,\cdots,\alpha_{k-1})} \int_{\alpha_{-1}\beta^{-1}+\alpha_0\beta^0+\cdots+\alpha_{k-1}\beta^{k-1}+P^k} |D_{q^k}(x)|dx \\ &= q^{-k} \sum_{(\alpha_{-1},\alpha_0,\ldots,\alpha_{k-1})} |D_{q^k}(\alpha_{-1}\beta^{-1} + \alpha_0\beta^0 + \cdots + \alpha_{k-1}\beta^{k-1})| \\ &\equiv q^{-k} I_k. \qquad (21)\end{aligned}$$

Consider the mapping $\Theta : (\alpha_{-1}, \alpha_0, \ldots, \alpha_{k-1}) \longmapsto (\Gamma^0, \Gamma^1, \ldots, \Gamma^{c-1})$ between $c(k+1)-$tuples , where

$$\begin{aligned} \alpha &\equiv (\alpha_{-1}, \alpha_0, \cdots, \alpha_{k-1}) \\ &= (\alpha_{-1,0}, \alpha_{-1,1}, \cdots, \alpha_{-1,c-1}; \alpha_{0,0}, \alpha_{0,1}, \cdots, \alpha_{0,c-1}; \cdots; \alpha_{k-1,0}, \alpha_{k-1,1}, \cdots, \alpha_{k-1,c-1}), \\ \gamma &\equiv (\gamma^0_{-1,0}, \gamma^0_{0,0}, \cdots, \gamma^0_{k-1,0}; \gamma^1_{-1,0}, \gamma^1_{0,0}, \cdots, \gamma^1_{k-1,0}; \cdots; \gamma^{c-1}_{-1,0}, \gamma^{c-1}_{0,0}, \cdots, \gamma^{c-1}_{k-1,0}) \end{aligned} \tag{22}$$

for $\alpha_{i,j}, \gamma^j_{i,0} \in GF(p)$. The mapping Θ is defined precisely by : if

$$x = \alpha_{-1}\beta^{-1} + \alpha_0\beta^0 + \cdots + \alpha_{k-1}\beta^{k-1}$$

and

$$\varepsilon_j x = \gamma^j_{-1}\beta^{-1} + \gamma^j_0\beta^0 + \cdots + \gamma^j_{k-1}\beta^{k-1} (mod P^k),$$

then we identify the image $\gamma = \Theta(\alpha)$ as the unique $\gamma^j_{i,0}$ ordered as the $c(k+1)-$tuples in (22).

If we can show that Θ is a 1-1 correspondence , we have by (21), (20)

$$\begin{aligned} I_k &= \sum_{(\alpha_{-1},\alpha_0,\ldots,\alpha_{k-1})} |D_{q^k}(\alpha_{-1}\beta^{-1} + \alpha_0\beta^0 + \cdots + \alpha_{k-1}\beta^{k-1})| \\ &= \sum_{(\Gamma^0,\Gamma^1,\ldots,\Gamma^{c-1})} \prod_{j=0}^{c-1} H_{\Gamma^j}. \end{aligned}$$

From the above discussion on cases $(A)-(E)$, we see for each j there are $2(k+1)$ cases such that $H_{\Gamma^j} \neq 0$, hence $(2(k+1))^c$cases for $\gamma = (\Gamma^0, \Gamma^1, \ldots, \Gamma^{c-1})$ such that $\prod_{j=0}^{c-1} H_{\Gamma^j} \neq 0$.

To compute the sum I_k we impose an arbitrary order for cases A, B_ℓ, C_ℓ, D; namely, $\sigma^1, \sigma^2, \ldots, \sigma^{2(k+1)} (\sigma^\ell = Case\ \ell)$. If $i \equiv (i(0), i(1), \ldots, i(c-1)), 1 \leq i(j) \leq 2(k+1)$, let $\sigma_i = \prod_{j=0}^{c-1} \sigma^{i(j)}$ mean case $\sigma^{i(j)}$ occurs for Γ^j, respectively (in which case write $\Gamma^j \in \sigma^{i(j)}$). Then $\{\sigma_i\}_{i=(i(0),i(1),\ldots,i(c-1))}$ are exactly the $2^c(k+1)^c$ cases where $\prod_{j=0}^{c-1} H_{\Gamma^j \in \sigma^{i(j)}} \neq 0$. Denote by $T^{i(j)}$ the number of tuples in $\Gamma^j \in \sigma^{i(j)}$ (e.g. the number of tuples in $\Gamma^j \in Case\ (B_1)$ is $(p-1)^2 p^{k-2}$), then we observe from the inequalities in cases $(A)-(D)$ that

$$\frac{2}{\pi}\left(\frac{p-1}{p}\right)^2 p^k \leq T^{i(j)} H_{\Gamma^j \in \sigma^{i(j)}} \leq \frac{p(p-1)}{2} p^k.$$

As a result we have

$$\begin{aligned} I_k &= \textstyle\sum_{(\Gamma^0,\Gamma^1,\ldots,\Gamma^{c-1})} \prod_{j=0}^{c-1} H_{\Gamma^j} = \sum_{\sigma_i = \prod_{j=0}^{c-1} \sigma^{i(j)}} (\prod_{j=0}^{c-1} T^{i(j)})(\prod_{j=0}^{c-1} H_{\Gamma^j \in \sigma^{i(j)}}) \\ &= \textstyle\sum_{i=(i(0),i(1),\ldots,i(c-1))} \prod_{j=0}^{c-1} (T^{i(j)} H_{\Gamma^j \in \sigma^{i(j)}}) = \sum_i \prod_{j=0}^{c-1} (A_j p^k) = 2^c A^c (k+1)^c (p^k)^c, \end{aligned}$$

for some A_j, A possibly depending on k, j with $\frac{2}{\pi}(\frac{p-1}{p})^2 \leq A_j, A \leq \frac{p(p-1)}{2}$.

Finally we conclude that if $Char K_q = 0$, then, noting (7),

$$\begin{aligned} \textstyle\int_{|x|=q} |D_{q^k}(x)| dx &= \textstyle\int_{|x|\leq q} - \int_{|x|\leq 1} \\ &= q^{-k} I_k - q^{-k} \textstyle\int_{P^k} dx = 2^c A^c (k+1)^c - 1 = A'^c k^c, \end{aligned}$$

where $A' \leq 2p^2$ for $k \geq 1$ and $A' \geq (p-1)^2 p^{-2}$ for $k = k(p)$ large enough. $\square$

To complete our proof, we need the following fact used in the above process for the p-adic case.

Lemma 8. The mapping Θ defined in (22):

$$(\alpha_{-1},\alpha_0,\cdots,\alpha_{k-1}) \longmapsto (\Gamma^0,\Gamma^1,\cdots,\Gamma^{c-1})$$

between $c(k+1)-$tuples is a one-to-one correspondence, where $\alpha_i = (\alpha_{i,0},\alpha_{i,1},\ldots,\alpha_{i,c-1})$ and $\Gamma^j = (\gamma^j_{-1,0},\gamma^j_{0,0},\ldots,\gamma^j_{k-1,0}),\ \alpha_{i,j},\gamma^j_{i,0} \in GF(p)$.

Proof. It suffices to show that Θ is injective. Suppose $\Theta(x)=\Theta(x')$, i.e., $\Gamma^j=\Gamma'^j$ for all j. We will prove $x=x'$. By the definition of Θ we have (for brevity we omit (mod P^k) in the following equalities) if $x=\sum_{i=-1}^{k-1}\alpha_i\beta^i$, from the last equation in (19):

$$\begin{aligned}\varepsilon_j x &= (\gamma^j_{-1}\beta^{-1}+\gamma^j_0\beta^0+\cdots+\gamma^j_{k-1}\beta^{k-1})\\ &= (\gamma^j_{-1,0}\beta^{-1}+\gamma^j_{0,0}\beta^0+\cdots+\gamma^j_{k-1,0}\beta^{k-1})\varepsilon_0+\\ &\quad (\gamma^j_{-1,1}\beta^{-1}+\gamma^j_{0,1}\beta^0+\cdots+\gamma^j_{k-1,1}\beta^{k-1})\varepsilon_1+\cdots+\\ &\quad (\gamma^j_{-1,c-1}\beta^{-1}+\gamma^j_{0,c-1}\beta^0+\cdots+\gamma^j_{k-1,c-1}\beta^{k-1})\varepsilon_{c-1};\end{aligned}$$

similarly, if $x'=\sum_{i=-1}^{k-1}\alpha'_i\beta^i$,

$$\begin{aligned}\varepsilon_j x' &= (\gamma'^j_{-1,0}\beta^{-1}+\gamma'^j_{0,0}\beta^0+\cdots+\gamma'^j_{k-1,0}\beta^{k-1})\varepsilon_0+\\ &\quad (\gamma'^j_{-1,1}\beta^{-1}+\gamma'^j_{0,1}\beta^0+\cdots+\gamma'^j_{k-1,1}\beta^{k-1})\varepsilon_1+\cdots+\\ &\quad (\gamma'^j_{-1,c-1}\beta^{-1}+\gamma'^j_{0,c-1}\beta^0+\cdots+\gamma'^j_{k-1,c-1}\beta^{k-1})\varepsilon_{c-1},\end{aligned}$$

where $\gamma'^j_{i,\ell}\in GF(p)$ are self-evident.

Since $\Gamma^j=\Gamma'^j$ for all j , we see

$$\varepsilon_j(x-x') = b^j_1(\beta)\varepsilon_1+\cdots+b^j_{c-1}(\beta)\varepsilon_{c-1}$$

for some Laurent polynomials $b^j_m(\beta)$ over the p-adic field K_p with indeterminate β, i.e., $b^j_m\in K_p$, $m=1,2,\ldots,c-1$.

Suppose $x-x'\neq 0$, then $\{\varepsilon_j(x-x')\}_{j=0}^{c-1}$ is linear independent in K_{p^c} over K_p. We would have

$$span\{\varepsilon_j(x-x')\}_{j=0}^{c-1} = span\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{c-1}\}\ (over K_p). \tag{23}$$

On the other hand

$$span\{b^j_1(\beta)\varepsilon_1+\cdots+b^j_{c-1}(\beta)\varepsilon_{c-1}\}_{j=0}^{c-1}\subset span\{\varepsilon_1,\ldots,\varepsilon_{c-1}\}\ (over\ K_p),$$

which contradicts (23). So we must have $x=x'$. This completes the proof. □

Remark. For $K_q=K_p(c=1)$ the p-adic field, we do not need this lemma, in which case, the mapping Θ reduces to an identity mapping and our estimate for the sum I_k becomes more straightforward.

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