Review Final **Name**
Math 142 **Section** **Id**

Use exactly one page for each of the numbered questions (use the back of the page if necessary).
Put your name and the question number on each page.
Put a box around the final answer to a question.
You must *show* your work in order to get possible credits.

1. Find the limit of the sequence

a) $\lim_{n\to\infty} \frac{\cos n}{\ln n}$

Hint: Since

$$|a_n| \leq \frac{1}{\ln n} \to 0$$

as $n \to \infty$, we find $\lim_{n\to\infty} |a_n| = 0$ which means $\lim_{n\to\infty} a_n = 0$.

b) $\lim_{n\to\infty} \frac{n^5-4n^3+7}{5n^4+n^2+100}$

Hint:

$$a_n = \frac{(n^5 - 4n^3 + 7)/n^4}{(5n^4 + n^2 + 100)/n^4} = \frac{n - \frac{4}{n} + \frac{7}{n^4}}{5 + \frac{1}{n^2} + \frac{100}{n^4}}$$

Since the top goes to ∞ and the bottom goes to 5, we must have $a_n \to \infty$.

c) $\lim_{n\to\infty}(1 - \frac{2}{3n})^n$

Hint:

$$a_n = e^{n\ln(1-\frac{2}{3n})}$$

Use L'hopital rule to show $n\ln(1 - \frac{2}{3n}) \to -2/3$ hence $a_n \to e^{-2/3}$.

d) $\lim_{n\to-\infty} \frac{e^n-1}{e^n+1}$

2. Determine whether the limit of the following function/sequence exists, if so, find the limit:

a) $\lim_{x\to 0+} \frac{\sin^{-1}\sqrt{x}}{\sqrt{x}}$

Hint: Sub $y = \sqrt{x}$ then

$$\lim \frac{\sin^{-1}\sqrt{x}}{\sqrt{x}} = \lim_{y\to 0} \frac{\sin^{-1} y}{y} = 1$$

by L'hopital rule.

b) $\lim_{x\to +\infty}(x\ln x)^2 e^{-x}$

Hint: Write

$$(x\ln x)^2 e^{-x} = \frac{(x\ln x)^2}{e^x}$$

then apply L'hopital a couple of times

c) $\lim_{x\to\infty} x\tan(\pi/x)$

d) $\lim_{n\to\infty} \frac{n!}{n^n}$

Hint: Instead of finding the limit of the sequence $a_n = \frac{n!}{n^n}$ directly, we consider the series $\sum_n a_n$. If we can show the series converges, then by the basic property of a convengent series we know that a_n must converge to zero.

To show the convergence of $\sum_n a_n$ we use ratio test (or root test if you prefer) to find that $\lim a_{n+1}/a_n = 0 < 1$, which means convergence of $\sum a_n$.

e) $\lim_{n\to\infty} 2^{-n}$

f) $\lim_{n\to\infty} (-2)^n$

Fill in the blanks or parenthesis in Problems **3** to **8**.

3. Let $a > 0$ be a constant. (a) $\int \frac{dx}{a^2+x^2}$ = ________ $+C$

(b) $\int \frac{dx}{a^2-x^2}$ = ________ $+C$

c) $\int \frac{dx}{\sqrt{a^2+x^2}}$ = ________ $+C$

d) $\int \frac{dx}{\sqrt{a^2-x^2}}$ = ________ $+C$

4 (a) If $a > 0$ but $a \neq 1$, then $D_x\ (a^x)$ = ______________________________
Hint: $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$.

(b) $D_x\ (x^x)$ = ______________________________ Hint: $x^x = e^{x\ln x}$, then apply chain rule.

5. Trig substitution: (recall that the *integrand* is the function you are integrating)
a) if the integrand involves $a^2 - u^2$, then one makes the substitution $u =$ ____________________
b) if the integrand involves $a^2 + u^2$, then one makes the substitution $u =$ ____________________

6. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do ________________________

7 (a) A series $\sum a_n$ is said to *converge absolutely* if $\sum |a_n|$ ________________
Give an example of an absolutely convergent series: ________________
(b) A series $\sum a_n$ is said to *converge conditionally* if $\sum a_n$ is ________________ but $\sum |a_n|$ __________________. Give an example of a conditionally convergent series: ________________

8 (a) Consider the interval $I = (a - R, a + R)$ center about $x = a$ and of radius R.
Let $y = f(x)$ be a function that can be differentiated N times $x = a$.
Then the N^{th}-order Taylor polynomial $y = P_N(x)$ of f about a is (your answer should have a summation sign $\sum$ in it)

$P_N(x) \;=$

(b) Consider the interval $I = (a - R, a + R)$ center about $x = a$ and of radius R.
Let $y = f(x)$ be a function that can be differentieated $N + 1$ times for each $x \in I$.
Consider the the N^{th}-order Taylor Reminder term $R_N(x)$, where $f(x) = P_N(x) + R_N(x)$.
Then an upper bound for $|R_N(x)|$ for an $x \in I$ is:

$\lvert R_N(x)\rvert \leq$

9. Use chain rule or logarithm derivative method to find the derivative. a) $D_x\ (\cos(\ln x))\ =$

b) $D_x\ (7^{(x-2)^2})\ =$

10. Evaluate the integrals. a) $\displaystyle\int (\tan x)\ \left(\sec^7 x\right)\ dx\ =$

Hint: Sub $u = \sec x$, then $du = \tan x \sec x dx$. $\int (\tan x)\ (\sec^7 x)\ dx = \int u^6 du$

b) $\displaystyle\int x^2 \arctan x\, dx\ =$

Hint: Integrating by parts, $u = \arctan x$, $dv = x^2 dx$ then $du = \frac{1}{1+x^2} dx$, $v = x^3/3$

$$\int x^2 \arctan x\, dx = uv - \int v du$$

c) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\ dx\ =$

d) $\int x^{\frac{1}{2}} \ln x\, dx$

Hint: Integration by parts

e) $\int_0^1 \frac{u^3}{(u+1)^2} du$

Method I. Partial fraction:

$$\frac{u^3}{(u+1)^2} = (u-2) + \frac{3u+2}{(u+1)^2} = u - 2 + \frac{3}{u+1} - \frac{1}{(u+1)^2}$$

Method II. Substitute $y = u + 1$.

f) $\int \frac{1-x}{1+x+x^2}\ dx$

Hint: The factor $1 + x + x^2$ is an irreducible quadratic polynomial, that is, it does not have real root. So the integrand is a rational function in a standard form already.

Complete the square $1 + x + x^2 = 3/4 + (x + 1/2)^2$ and write

$$\frac{1-x}{1+x+x^2} = -\frac{x+1/2}{3/4+(x+1/2)^2} + \frac{3/2}{3/4+(x+1/2)^2}$$

g) $\int x(\sqrt{x}+1)^{\frac{1}{3}} dx$

11. Let R be the region enclosed by $y = x^2$, $x = 2$ and $y = 0$. Let V be the volume of the solid obtained by revolving the region R about the line $x = 3$.

(a) Make a rough sketch below of the region R, labeling the important points.

(b) Using the disk/washer method, express the volume V as an integral (or maybe 2 integrals). You do NOT have to evaluate the integral(s).

V =

12. Determine whether the improper integral converges.

$$(a) \quad \int_0^1 \ln(1-x)\, dx$$

Hint: $x = 1$ is a singularity. Consider $\int_0^b \ln(1-x)dx$ as $b \to 1_-$.

$$(b) \quad \int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\, dx$$

Hint: $x = 0$ is a singularity. The inequality

$$|\frac{\cos x}{\sqrt{x}}| \leq \frac{1}{\sqrt{x}}$$

and the fact that $\int_0^{\pi/2} \frac{dx}{\sqrt{x}} < \infty$ suggests that the original integral converges by direct comparison test for improper integrals.

$$(c) \int_0^{\infty} \frac{dx}{5000+x}$$

Hint:

$$\lim_{x\to\infty} \frac{\frac{1}{5000+x}}{\frac{1}{x}} = 1 \neq 0$$

By limit comparison test we know the original integral diverges because $\int 1/x dx = \infty$.

13. Does the following series converge or diverge? Explain your answer by stating which kind of test you are using and how it works.

$$a. \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

Hint:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = 1/2 \neq 0$$

By limit comparsion test the original series diverges because the p-series $\sum 1/\sqrt{n} = \infty$.

$$b^*(optional) \quad \sum_{n=0}^{\infty}(-1)^n \frac{1}{(n+1)^{0.6}}$$

$$c) \quad \sum_{n=2}^{\infty} ne^{-\sqrt{n}}$$

Hint: Method I. Integral test. Consider $\int_2^\infty xe^{-\sqrt{x}}dx$.
Method II. Limit comparison test. Since

$$\lim_n \frac{ne^{-\sqrt{n}}}{1/n^2} = 0 < \infty$$

by L.C.T. the original series converges because the p-seires $\sum 1/n^2 < \infty$.

Remark. ratio test D.N. work because $\lim_n a_{n+1}/a_n = 1$, which does not yield a conclusion.

14. Determine the radius and interval of convergence of the power series

$$a) \quad \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Ans: radius $R = 1$; interval $[-1, 1)$

$$b) \quad \sum_{n=2}^{\infty} (\ln n) x^{2n+1}$$

Ans: radius $R = 1$; interval $(-1, 1)$

15. Let $a_n = \frac{e^n \cdot n!}{(2n)!}$ Find $\frac{a_{n+1}}{a_n}$. Simplify your answer so that no factorial sign (i.e., !) appears.

answer: $\dfrac{a_{n+1}}{a_n} =$

$$\sum_{n=1}^{\infty} (-1)^n \, \frac{e^n \, (n!)}{(2n)!}$$

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

16. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2-x)^{n-1}}{5^n} \, .$$

In the box below draw a diagram indicating for which x's this series is: absolutely convergent, conditionally convergent, and divergent. Of course, indicate your reasoning.

17. Let

$$f(x) = (1+x)^{3/2}$$

Find the 3rd-order Taylor polynomial of $y = f(x)$ about $x = 0$.

$P_3(x) \;=\; 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{3}{48}x^3$

18. Find the Taylor or Maclaurin series of $y = f(x)$

(a)

$$f(x) = e^{2x+1}$$

about $x = 1$

Hint:

$$e^{2x+1} = e^{2(x-1)+3} = e^3 \sum_{n=0}^{\infty} \frac{2^n}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{2^n e^3}{n!}(x-1)^n, \quad -\infty < x < \infty$$

(b)

$$f(x) = \frac{1}{1+x^2}$$

about $x = 0$.

Hint:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1$$

19. The equation of the ellips is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Express the length of the ellips as an definite integral. Do not evaluate the integral.