MATH 1441, Review Test 1
Show your work

Put a box around the final answer to a question.
Show your work in order to get possible credits or partial credits.

1 [10P]) Write the equation for the tangent line and the normal line to the curve $f(x) = x^2 + 3x + 1$ at the point $P(1,5)$:

Tangent line:

Normal line:

2 [32P]) Find the following limits or determine if they do not exist:

a) $\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) =$

b) $\lim_{x\to 0} \sqrt{x^2+7x} - x =$

c) $\lim_{x\to 2} \frac{x^2+2x-3}{x^2+x-6} =$

d) $\lim_{x\to 3} \frac{|x+1|}{[[x+1]]} =$

e) $\lim_{x\to 5^-} \frac{1.5}{10-2x} =$

f) $\lim_{x\to \infty} \frac{x^2+1}{1-2x^2} =$

g) $\lim_{x\to -\infty} \frac{\cos x}{x} =$

3 [6P]) a) Explain if the equation $x^3 + x + 1 = 0$ has a solution in the interval $[-1, 0]$ or not.

b) Use the intermediate value theorem to show that each polynomial has a real zero between the given integers.

i) $x^3 - 4x^2 + 2$ between 0 and 1 ii) $x^4 + 6x^3 - 18x^2$ between 2 and 3 iii) $x^5 - x^3 - 1$ between 1 and 2

4 [11P]) Prove using the $\varepsilon - \delta$ definition of limit that

$$\lim_{x\to 2} \quad x - 3 = -1$$

5 [11P]) Let $f(x) = \frac{1-x}{1-x^2}$. Where is $f(x)$ defined? Find the left hand side and the right hand side limits at the points where $f(x)$ is not defined. Is it possible to assign a value to $f(x)$ at those points such that $f(x)$ is continuous at the point?

6 [28P]) Find the derivative of the following functions:

a) $f(x) = \sqrt{x^2-1} + x^2$. $f'(x) =$

b) $f(x) = \frac{x^3+3x}{x+1}$. $f'(x) =$

c) $f(x) = \dfrac{1}{3x-5}$. $f'(x) =$

d) $f(x) = \cos(x) + \dfrac{2}{x}$ $\quad f'(x) =$

e) $f(x) = \left(\dfrac{1}{x^2+1}\right)^2$ $\quad f'(x) =$

f) The curve of $y = y(x)$ is given by the parametric equation $x = 5\cos t + 5, y = 3\sin t - 3$ Find the derivative dy/dx at the point (5,0), that is, at $t = \pi/2$.

g) Do the same as in (f) for the curve $x = at$, $y = a(1 - \cos t)$ at $t = \pi$ and $t = 3\pi$.

7 [20P]) Sketch the graph of the function $f(x) = \dfrac{x^2}{x+1}$. Identify and label all extrema ([4P]), inflection points ([4P]), intercepts ([4P]), and asymptotes ([4P]). Indicate the concave structure clearly ([4P]).

8[11P] A water bucket containing 10 gal of water develops a leak at time $t = 0$. The volume V of water in the bucket t seconds later is given by

$$V(t) = 10\left(1 - \frac{t}{100}\right)^2$$

until the bucket is empty at time $t = 100$. a) At what rate is water leaking from the bucket after exactly 1 min. b) What is the average rate of change of V from time $t = 0$ and $t = 50$ and from time $t = 0$ and $t = 100$?

9[10P] States the Sandwich theorem (or Squeezing Theorem) on the limit rule.

10*[11P] Is the following function a) continuous at 0 ? b) differentiable at 0?

$$g(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

11*[9P] Determine whether it is possible to choose constants c_1, c_2, c_3 so that the function $h(x)$ is (a) continuous (b) differentiable. If so, find the constants.

$$h(x) = \begin{cases} c_1 \tan x & -\pi/2 < x < -\pi/4 \\ c_2|x| & -\pi/4 \le x \le \pi/4 \\ c_3 \tan x & \pi/4 < x < \pi/2 \end{cases}$$

12[20P] Use implicit differentiation to find dy/dx.

$$a)\ x^{3/2} + y^{3/2} = 1$$

$$b)\ y\cos x + x\sin y = \tan(y/x)$$

Answer to Review Test 1

2. a) 0 Use Sandwich Theorem by noting $-|x| \leq |x \sin(1/x^2)| \leq |x|$

d) $\lim_{x\to 3+}[[x+1]] = 4$, $\lim_{x\to 3-}[[x+1]] = 3$ $\therefore$ $\lim_{x\to 3+} \frac{|x+1|}{[[x+1]]} = 1$, $\lim_{x\to 3-} \frac{|x+1|}{[[x+1]]} = 4/3$. Limit does not exist.

e) $+\infty$ g) 0

3. Let $f(x) = x^3 + x + 1$. $f(0) = 1$, $f(-1) = -1$. Since f is continuous, by intermediate value theorem, on $(-1, 0)$ f takes all values between -1 and 1. In particular $f(c) = 0$ for some $c \in (-1, 0)$.

4. Proof. Given $\epsilon > 0$. Need to find $\delta = \delta(\epsilon)$

$$(*) \qquad |f(x) - L| < \epsilon \qquad \textit{whenever } |x - 2| < \delta.$$

Put $f(x) = x - 3$, $L = -1$ in $(*)$ and find that we can choose $\delta = \epsilon$ so that $(*)$ holds trivially.

6. a) By Chain rule

$$f'(x) = \frac{1}{2}(x^2-1)^{-1/2} \cdot (x^2-1)' + 2x$$
$$= \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x + 2x = \frac{x}{(x^2-1)^{1/2}} + 2x$$

c) $-\frac{3}{(3x-5)^2}$ d) $f'(x) = -\sin x - 2/x^2$

10. g is continuous but not differentiable at 0.

11. a) To make f continuous at $x = \frac{\pi}{4}$, need $\lim_{x\to\pi/4+} f(x) = \lim_{x\to\pi/4-} f(x)$, which means $c_3 = c_2\frac{\pi}{4}$; similarly to make f continuous at $x = -\frac{\pi}{4}$, need $\lim_{x\to-\pi/4+} f(x) = \lim_{x\to-\pi/4-} f(x)$, which requires $-c_1 = c_2\frac{\pi}{4}$.

b) To make f differentiable at $x = \frac{\pi}{4}$, need $f'(x)|_{\frac{\pi}{4}+} = f'(x)|_{\frac{\pi}{4}-}$, which require $c_2 = 2c_3$. Similarly, at $x = -\pi/4$, we will require $2c_1 = -c_2$.