MATH 1441, Review Test # 2

Show all your work in order to get possible credits or partial credits.

1 Find the derivatives of the following functions:

a) $f(x) = \cos(3x) + \dfrac{2}{\sqrt{x+10}}$ $\quad f'(x) =$

b) $f(x) = \sec(\tan x)$ $\quad f'(x) =$

c) $y = \sin^{-1}\left(\sqrt{1-\sqrt{t}}\right)$, $\quad (0 < t < 1)$ $\quad dy/dt =$

d) $y = (1 + \dfrac{1}{x})^3$ $\quad d^2y/dx^2 =$ (EX 3.5, Prob. 49)

2 a) Find the slope of the tangent line to the graph of the equation $xy^3 + x^2y = 10$ at the point $(1, 2)$. The slope is:

b) Find an equation for the line tangent to the curve at the point defined by the given value at $t = \pi/3$. Also find the value of d^2y/dx^2 at this point.

$$x = t - \sin t,\ y = 1 - \cos t.$$

(hint: EX 3.5, Prob. 92)

3 a) Explain if the equation $x^3 + x + 1$ has a solution in the interval $[-1, 0]$ or not. (Hint for a) Intermediate Value Theorem)

Show the following functions have exactly one zero in the given interval.

b) $f(x) = x^4 + 3x + 1$, $[-2, -1]$ (EX 4.2, Prob. 15)

c) $r(\theta) = \tan\theta - \cot\theta - \theta$, $(0, \pi/2)$ (EX 4.2, Prob. 22)

(Hints for b), c) Rolle's Theorem)

4 Find the absolute maximum and minimum values of each function on the given interval. Indicate on the graph where the extrema occur and where the graph increases and/or decreases. a) $f(x) = -3x^{2/3}$, $[-1, 1]$

b) $g(x) = x^4 - 2x^2 + 1$, $[-2, 2]$.

c) $h(x) = \frac{x+1}{x^2+2x+2}$, $(-\infty, \infty)$

5 Sketch the graph of the function. Identify and label all extrema ([4P]), inflection points ([4P]), intercepts ([4P]), and asymptotes ([4P]). Indicate the concave structure clearly ([4P]).

a) $f(x) = \dfrac{x^2}{x+1}$.

b) $g(x) = x^4 - 2x^2$ (EX 4.4, Prob.17)

c) $h(x) = x + \sin x$, $0 \le x \le \pi$ (EX 4.4, Prob.23)

6 Find the limits. Use the l'Hospital's Rule where applicable. If l'Hospital's Rule doesn't apply, explain why.

a) $\lim_{x\to -2} \dfrac{x+2}{x^2+3x+2}$

b) $\lim_{x\to(\pi/2)^+} \dfrac{1-\sin x}{\cos x}$

d) $\lim_{x\to\infty} x\tan(1/x)$

e) $\lim_{x\to\infty}(x-\sqrt{x^2+x})$ (EX 4.6, Prob. 23)

f*) $\lim_{x\to 0}(\cos x)^{1/x^2}$

g) $\lim_{x\to 0^+} \dfrac{\sqrt{7x}}{\sqrt{\sin 3x}}$

h) $\lim_{x\to 0}(\dfrac{1}{x}-\dfrac{1}{\sqrt[3]{x}})$

7 Find the most general form of antiderivative or indefinite integral.

a) $\displaystyle\int \frac{2x^4-3x+5}{x^5}\,dx =$

c) $\displaystyle\int x(x-1)^2\,dx =$

d) $\displaystyle\int 2x^2\sin(x^3)\,dx.$

e) $\displaystyle\int \sec\frac{\pi x}{2}\tan\frac{\pi x}{2}$ (EX 4.8, Prob.16(c))

f) $\displaystyle\int 2x(1-x^{-3})dx$ (EX 4.8, Prob.31)

g) $\displaystyle\int \cos\theta(\tan\theta+\sec\theta)d\theta$ (EX 4.8, Prob.53)

8. Solve the initial value problems. a) $\dfrac{dy}{dx}=2x-7,\ y(2)=0$

b) $\dfrac{dv}{dt}=8t+\csc^2 t,\ v(\dfrac{\pi}{2})=-1.5$

9 a) Give the definition of Linearization or standard Linear Approximation. Let f be differentiable at $x=a$. The linearization of f is given by ____________

b) The linearization of $g(x)=\sqrt{4+x}$ at $x=0$ is ____________

c) The linearization of $h(x)=\tan x$ at $x=\pi/4$ is ____________

(EX 3.8, Prob. 14(b))

10* (Optimization) An open-top box is to be made by cutting small congruent squares from the corners of a 12-in by 12-in sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

11* (optional) Water runs into a conical tank at the rate of $9ft^3/min$. The tank stands point down and has a height of 10 ft and a base radius of 5ft. How fast is the water level rising when the water is $6ft$ deep?

12 A boat leaves a dock at 2:00 p.m. and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 p.m. At what time were the two boats closest together?

Answer to Review Test 2

1. a)

$$\frac{d}{dx}[\cos(3x) + \frac{2}{\sqrt{x+10}}] = \frac{d}{dx}[\cos(3x)] + \frac{d}{dx}[\frac{2}{\sqrt{x+10}}]$$

$$= -\sin(3x)\cdot 3 + 2(-1/2)(x+10)^{-3/2} = -3\sin(3x) - \frac{1}{(x+10)^{3/2}}$$

b) By chain rule

$$\frac{d}{dx}[\sec(\tan x)] = \sec(\tan x)\tan(\tan x)\cdot(\sec^2 x)$$

3. a) Let $f(x) = x^3 + x + 1$. Since $f(-1) = -1 < 0, f(0) = 1 > 0$, according to intermediate value Theorem, there must exists some point c in $[-1, 0]$ such that $f(c) = 0$ which is between $f(-1)$ and $f(0)$.

b) By intermediate value theorem, since $f(-2) = 11 > 0$, $f(-1) = -1 < 0$, we know there exist some c in $[-2, -1]$ such that $f(c) = 0$.

On the other hand $f'(x) = 4x^3 + 3$, on $[-2, -1]$ which is less than $4(-1)^3 + 3 = -1 < 0$, so the function f(x) decreases on $[-2, -1]$. This suggests that f(x) can have only one zero on the interval.

4. a) $y'(x) = -2x^{-1/3}$ Critical point: $x = 0$ (critical points are those where derivative vanishes or does not exist)

$f(-1) = f(1) = -3$, $f(0) = 0$ $\quad \therefore y_{max} = 0$, $y_{min} = -3$.

Increasing on $[-1, 0]$, decreasing $[0, 1]$.

c)

$$h'(x) = -\frac{x(x+2)}{(x^2+2x+2)^2}$$

Solve $h'(x) = 0 \Longrightarrow$ Critical points are $x = 0, -2$

x		-2		0	
h'	$-$	0	$+$	0	$-$
h	$\searrow$		$\nearrow$		$\searrow$
h(x)		$-1/2$		$1/2$	

Vertical asymptote: None; horizontal asymptote: $y = 0$

x- intercept: -1; y- intercept: $1/2$

Hence absolute maximum $= 1/2$, absolute minimum $= -1/2$

5 a)

$$y'(x) = \frac{x(x+2)}{(x+1)^2}$$

$$y''(x) = \frac{2}{(x+1)^3}$$

Vertical asymptote: $x = -1$; horizontal asymptote: None; slant asymptote: $y = x - 1$

x- intercept: -1; y- intercept: $1/2$

Critical points are $x = 0, -2$

Inflection point: None

x		-2		-1		0	
y'	$+$	0	$-$		$-$	0	$+$
y	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$
y''	$-$				$+$		
y	concave down				concave up		
y(x)		-4				0	

Local maximum $y = -4$ at $x = -2$; local minimum $y = 0$ at $x = 0$

6 a) 0/0 type, can apply L.H. rule to get

$$\lim_{x \to -2} \frac{x+2}{x^2+3x+2} = \lim_{x \to -2} \frac{(x+2)'}{(x^2+3x+2)'} = \lim_{x \to -2} \frac{1}{2x+3} = \frac{1}{2(-2)+3} = -1$$

b) 0/0 type; applying L'hospital Rule we get

$$\lim_{x \to (\pi/2)^+} = \lim_{x \to (\pi/2)^+} \frac{-\cos x}{-\sin x} = \lim_{x \to (\pi/2)^+} \frac{-0}{-1} = 0$$

7 f)

$$\int (2x - 2x^{-2}) dx = 2 \int x dx - 2 \int x^{-2} dx = x^2 + \frac{2}{x} + C$$

11. Let $h = h(t)$ the depth of water (in feet) in the conical tank at t minutes and let $r = r(t)$ be the corresponding radius at that time t. Then the volume of water is

$$V(t) = \frac{1}{3} h(t) \cdot \pi r(t)^2$$

By similarity between two right triangles

$$\frac{r(t)}{h(t)} = \frac{5}{10}.$$

We have

$$V(t) = \frac{\pi}{12} h(t)^3 = 9t$$

Differentiating the equation with respect to t we get

$$\frac{\pi}{4} h(t)^2 \frac{dh}{dt} = 9 \Longrightarrow \frac{dh}{dt} = \frac{36}{\pi h^2}$$

Hence when h is 6 ft in depth the raising speed of water $= 1/\pi$ (ft/min).

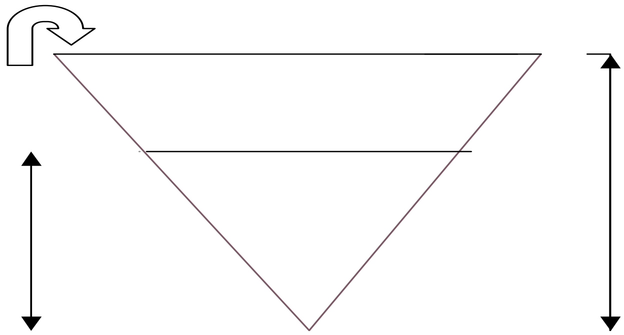

FIGURE 1. The conical tank with height h=10 ft, base radius r=5 ft