§*2.6 Limits Involving Infinity; Asymptotes of Graphs* (Continued)

Example 2. Find (a) $\lim_{x\to\infty}\left(5+\frac{1}{x}\right)$ (b) $\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2}$.

Example 3. Find (a) $\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}$.

Example 9. Find $\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)$.

Example 11. Find $\lim_{x\to 1^+}\frac{1}{x-1}$ and $\lim_{x\to 1^-}\frac{1}{x-1}$.

Example 7. Find $\lim_{x\to 0^-} e^{1/x}$.

Example 13. Find (c) $\lim_{x\to 2^+}\frac{x-3}{x^2-4}$ (d) $\lim_{x\to 2^-}\frac{x-3}{x^2-4}$ (f) $\lim_{x\to 2}\frac{2-x}{(x-2)^3}$.

Example 14. Find $\lim_{x\to-\infty}\frac{2x^5-6x^4+1}{3x^2+x-7}$.

§*3.1 Tangent Lines and the Derivative at a Point*

Definition. The **slope of the curve** $y=f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \qquad \text{(provided the limit existis).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example 1. (a) Find the slope of the curve $y=1/x$ at any point $x=a\neq 0$. What is the slope at the point $x=-1$?
(b) Where does the slope equal $-1/4$?

Definition. The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h},$$

provided this limit exists.

[Ex. from MML]

• (1) Find the slope of the tangent line of $y = f(x) = \frac{1}{x-3}$ at $x = 6$.
(2) Find the tangent line equation of $y = f(x)$ at $x = 6$.

Solution. (1) Step 1. Calculate the derivative of $f'(x) = \lim_{h\to 0} \frac{\frac{1}{(x+h)-3} - \frac{1}{x-3}}{h}$. Simplify

$$\frac{1}{(x+h)-3} - \frac{1}{x-3} = \frac{(x-3)-(x+h-3)}{((x+h)-3)(x-3)} = \frac{-h}{((x+h)-3)(x-3)}$$
$$\Rightarrow \frac{\frac{1}{(x+h)-3} - \frac{1}{x-3}}{h} = \frac{-1}{((x+h)-3)(x-3)}.$$

Hence,

$$f'(x) = \lim_{h\to 0} \frac{-1}{(x+h)-3)(x-3)} = \frac{-1}{(x+0)-3)(x-3)} = \frac{-1}{(x-3)^2}.$$

Step 2. The slope m of the tangent line at $x = 6$ equals to $f'(6)$.
We have $m = f'(6) = \frac{-1}{(6-3)^2} = -\frac{1}{9}$.

(2) To find the tangent line equation we use the point-slope form $y - y_1 = m(x - x_1)$. Since at the point $(6, f(6)) = (6, \frac{1}{3})$, $m = -\frac{1}{9}$, we see that the equation of the tangent line is given by

$$y - \frac{1}{3} = -\frac{1}{9}(x-6)$$
$$Simplify \qquad y = -\frac{1}{9}x + 1.$$

□

§*3.2 The Derivative as a Function*

Definition. The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.
If f' exists at a particular x, we say that f is **differentiable at x**. If f' exists at every point in the domain of f, we call f **differentiable**.
The process of calculating a derivative is called **differentiation**.

Example 1. Differentiate $f(x) = \dfrac{x}{x-1}$.

Example 2. (a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

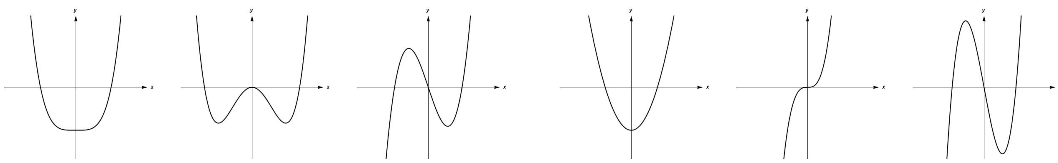
Notation. Some common alternative notations for the derivative include

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

Problem. Match the functions (a)–(c) with their derivatives (1)–(3).



(a) (b) (c) (1) (2) (3)

Example 4. Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$ but has no derivative at $x = 0$.

Theorem 1 (Differentiability Implies Continuity). If f has a derivative at $x = c$, then f is continuous at $x = c$.

<u>*§3.3 Differentiation Rules*</u>

Derivative of a Constant Function. If f has the constant value $f(x) = c$, then $\frac{df}{dx} = \frac{d}{dx}(c) = 0$.

Power Rule. If n is any real number, then $\frac{d}{dx}x^n = nx^{n-1}$, for all x where the powers x^n and x^{n-1} are defined.

Example 1. Differentiate the following powers of x.

(a) x^3

(b) $x^{2/3}$ $\qquad \frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{-1/3}.$

(d) $\frac{1}{x^4}$ $\qquad \frac{d}{dx}(x^{-4}) = -4x^{-5}.$

(f) $\sqrt{x^{2+\pi}}$ $\qquad \frac{d}{dx}(x^{1+\frac{\pi}{2}}) = (1+\frac{\pi}{2})x^{\frac{\pi}{2}}.$

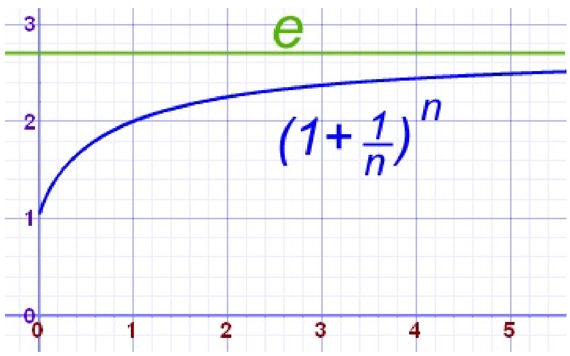
(z) x^e $\qquad \frac{d}{dx}x^e = ex^{e-1}.$

(The nature number $e \doteq 2.71828182845904523536028747135266549$ was studied by Leonhard Euler (1707-1783))

TABLE 1. e is the limit of a sequence of numbers in the table

n	$(1+\frac{1}{n})^n$
1	2.00000
2	2.25000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827

Try $n = 1,000,000,000$ in the calculator



Ex. From MML