§*3.3 Differentiation Rules* (Continued)

Derivative Constant Multiple Rule. If u is a differentiable function of x, and c is a constant, then $\frac{d}{dx}(cu) = c\frac{du}{dx}$.

Derivative Sum Rule. If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points, $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$.

Example 3. Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

[Answer:] $y' = \frac{dy}{dx} = 3x^2 + \frac{8}{3}x - 5$.

Example 4. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangent lines? If so, where?

Solution. Step 1. Evaluate the derivative $y' = 4x^3 - 4x = 4x(x - 1)(x + 1)$.

Step 2. Solve $y'(x) = 4x(x - 1)(x + 1) = 0 \Rightarrow x = 0, x = 1$ and $x = -1$. These are the points where the curve has zero slope for the tangent line, that is, where the curve has horizontal lines. □

Derivative of the Natural Exponential Function. $\frac{d}{dx}(e^x) = e^x$

Example 5*. Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

Solution. Let (a, b) with $b = e^a$ be a point on the curve with the required tangent line. Use the point-slope form for the tangent line: $y - b = m(x - a)$. Since $m = f'(a) = e^x|_{x=a} = e^a$, we have

$$y - e^a = e^a(x - a).$$

Now since the origin is on the line, $(0, 0)$ satisfies the equation:

$$0 - e^a = e^a(0 - a) \Rightarrow a = 1.$$

Hence the equation of the tangent line is given by $y - e = e(x - 1)$, or, in simplified form $y = ex$. □

Derivative Product Rule. If u and v are differentiable at x, then so is their product uv, and $\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$.

Example 6. Find the derivative of (a) $y = \frac{1}{x}(x^2 + e^x)$.

[Ans: $y'(x) = 1 + x^{-1}e^x - x^{-2}e^x$]

Derivative Quotient Rule. If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

Example 7. Find the derivative of (a) $y = \dfrac{t^2-1}{t^3+1}$, (b) $y = e^{-x}$.

[Ans: (a) $y'(t) = -\frac{t(t^3-3t-2)}{(t^3+1)^2} = -\frac{t(t+1)^2(t-2)}{(t+1)^2(t^2-t+1)^2} = -\frac{t(t-2)}{(t^2-t+1)^2}$, $\quad t \neq -1$
(b) $y'(x) = -e^{-x}$.]

Ex. (from homework/study guide)

(a) Evaluate the derivative of $f(x) = \frac{x+5}{x-3}$ for any $x \in (-\infty, 3) \cup (3, \infty)$
(b) Evaluate the derivative of $y = e^{2x} - 8e^{-x}$ at $x = 0, 1, -2$.
(c) Evaluate the derivate of $y = \frac{x}{x^2+1}$ at $x = 0$.

[Ans: (a) $f'(x) = -\frac{8}{(x-3)^2}$]

Definition. The **second derivative** of f is defined as $(f')'$, and is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

The ***n*th derivative** of y with respect to x for any positive integer n is denoted

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny.$$

Example 9. Find the first four derivative of $y = x^3 - 3x^2 + 2$.

Ex. From MML

§3.5 Derivatives of Trigonometric Functions

Derivative of Sine Function. $\dfrac{d}{dx}(\sin x) = \cos x$.

Proof. We will need $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ and $\lim\limits_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta} = 0$; also need the trigonometric identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x.$$

□

Example 1. Find the derivative: (b) $y = e^x \sin x$

[Answer: $y'(x) = e^x(\sin x + \cos x)$]

Derivative of Cosine Function. $\dfrac{d}{dx}(\cos x) = -\sin x$.

Example 2. Find the derivative: (c) $y = \dfrac{\cos x}{1 - \sin x}$

[Answer: $y' = \frac{1}{1-\sin x}$]

Derivatives of Other Trigonometric Functions.

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 5. Find $d(\tan x)/dx$.

Exercise 62. Derive the formula for the derivative with respect to x of: (a) $\sec x$

Example 6. Find y'' if $y = \sec x$.

§3.6 The Chain Rule

Example 1. The function $y = (3x^2+1)^2$ is obtained by composing the functions $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculate dy/dx.

Theorem 2 (The Chain Rule). If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$, where dy/du is evaluated at $u = g(x)$.

Exercise 82. Find the value of $(f \circ g)'$ at $x = -1$ for $f(u) = 1 - \dfrac{1}{u}$ and $u = g(x) = \dfrac{1}{1-x}$.

Example 3. Differentiate $\sin(x^2 + e^x)$ with respect to x.

Example 4. Differentiate $y = e^{\cos x}$.

Example 5. Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Example 8. Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive.

Ex. From MML