Math 1441 (Calc I) Section 4.1 Instructor: Dr. S. Zheng
Course website

§4.1 Extreme Values of Functions on Closed Intervals

Definition. Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if $f(x) \le f(c)$ for all x in D, and an **absolute minimum** value on D at c if $f(x) \ge f(c)$ for all x in D. Absolute maxima or minima are also referred to as **global** maxima or minima.

Example 1. Consider the defining equation $y = x^2$ on various domains:
(a) $D = (-\infty, \infty)$ (b) $D = [0, 2]$ (c) $D = (0, 2]$ (d) $D = (0, 2)$

Theorem 1 (The Extreme Value Theorem). If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

Ex. The function (a) $y = x^2$, with domain $[-1, 1]$; and (b) $y = x^3$ with domain $[-5, 5]$ are continuous functions that verify Theorem 1.

Ex. Let $f(x) = \begin{cases} x^2 & x \in [-1, 0) \cup (0, 1] \\ \frac{1}{2} & x = 0. \end{cases}$ Then $f(x)$ has no absolute minimum on $[-1, 1]$.

Ex. Let $g(x) = \begin{cases} x & x \in [-\frac{1}{2}, 0) \\ x - 1 & x \in [0, \frac{1}{2}]. \end{cases}$ Then $g(x)$ has no absolute maximum on $[-\frac{1}{2}, \frac{1}{2}]$.

Definition. A function f has a **local maximum** value at a point c within its domain D if $f(x) \le f(c)$ for all x in D lying in some open interval containing c. A function f has a **local minimum** value at a point c within its domain D if $f(x) \ge f(c)$ for all x in D lying in some open interval containing c. Local extrema are also called **relative extrema**.

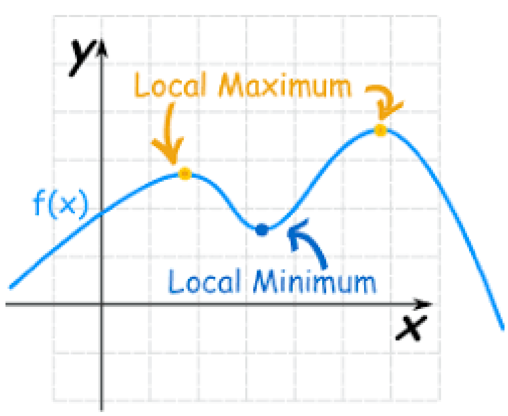

FIGURE 1. local extrema for $y = f(x)$ on its domain (courtesy: Math is fun)

Theorem 2 (The First Derivative Theorem for Local Extreme Values). If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof. We may assume f attains its local minimum at $x = c$.

$$f'(c) = \lim_{x \to c_+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c_+} \frac{(+)}{(+)} \geq 0.$$

On the other hand,

$$f'(c) = \lim_{x \to c_-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c_-} \frac{(+)}{(-)} \leq 0.$$

Hence, $0 \leq f'(c) \leq 0 \Rightarrow f'(c) = 0$. □

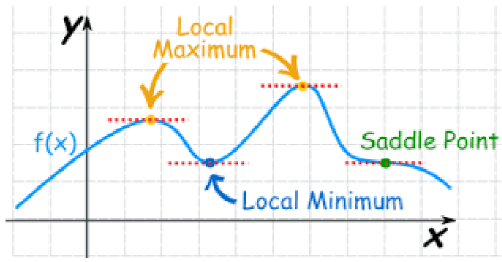

FIGURE 2. local extrema and saddle point for $f(x)$ on an interval (courtesy: Math is fun)

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