

§4.1 Extreme Values of Functions on Closed Intervals (Continued)

**Definition.** An interior point of the domain of a function  $f$  where  $f'$  is zero or undefined is a **critical point** of  $f$ .

Ex. Find all critical points of  $f(x) = x^3 - 3x^2 + 2$ .

*Solution.*  $f'(x) = 3x^2 - 6x$ . To find the critical point(s), we solve  $3x^2 - 6x = 0 \Rightarrow x = 0$  and  $x = 2$ . Therefore,  $x = 0, 2$  are the critical points of the function  $y = f(x)$  on  $\mathbb{R} = (-\infty, \infty)$ .  $\square$

**Exercise 52.** Determine all critical points for  $g(x) = \sqrt{2x - x^2}$ .

*Solution.*  $g'(x) = \frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x)$ . Solve  $\frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x) = 0$  to obtain  $2 - 2x = 0$ , that is,  $x = 1$ , which is the critical point in the interior of the domain  $[0, 2]$ .  $\square$

**Example 3.** Find the absolute maximum and minimum values of  $f(x) = 10x(2 - \ln x)$  on the interval  $[1, e^2]$ .

**Example 4.** Find the absolute maximum and minimum values of  $f(x) = x^{2/3}$  on the interval  $[-2, 3]$ .

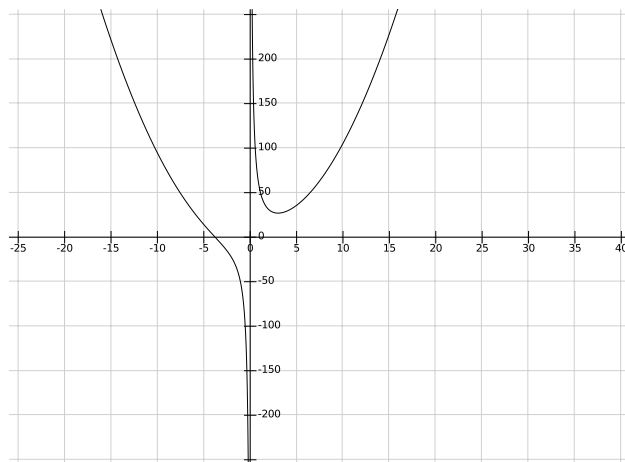
[Answer:  $x = 0$  where the derivative d.n.e.]

Ex. [MML](#) Determine the critical point(s) of  $y = x^2 + \frac{54}{x}$ .

[Answer:  $x = 3$ ]

§4.2 The Mean Value Theorem

**Theorem 3 (Rolle's Theorem).** Suppose that  $y = f(x)$  is continuous over the closed interval  $[a, b]$  and differentiable at every point of its interior  $(a, b)$ . If  $f(a) = f(b)$ , then there is at least one number  $c$  in  $(a, b)$  at which  $f'(c) = 0$ .



**Theorem 4 (The Mean Value Theorem).** Suppose  $y = f(x)$  is continuous over a closed interval  $[a, b]$  and differentiable on the interval's interior  $(a, b)$ . Then there is at least one point  $c$  in  $(a, b)$  at which 
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

**Corollary 1.** If  $f'(x) = 0$  at each point  $x$  of an open interval  $(a, b)$ , then  $f(x) = C$  for all  $x$  in  $(a, b)$ , where  $C$  is a constant.

**Corollary 2.** If  $f'(x) = g'(x)$  at each point  $x$  in an open interval  $(a, b)$ , then there exists a constant  $C$  such that  $f(x) = g(x) + C$  for all  $x$  in  $(a, b)$ . That is,  $f - g$  is a constant function on  $(a, b)$ .

Ex. From [MML §4.2](#): Let  $f(x) = \cos^{-1} x$ . Find value(s) of  $c \in [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$  s.t. 
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

[Solution] By the derivative formula for the inverse of cosine function, see §3.9 on [inverse trigonometric functions](#). 
$$\frac{d}{dx} (\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}.$$
 Identify  $a = -\frac{\sqrt{3}}{2}$ ,  $b = \frac{\sqrt{3}}{2}$ , and  $f(a) = \frac{5\pi}{6}$ ,  $f(b) = \frac{\pi}{6}$ . We substitute into the mean value theorem formula to solve for  $c = \pm\sqrt{1 - \frac{27}{4\pi^2}}$ .

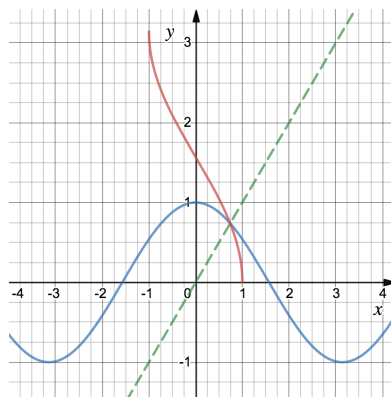


FIGURE 1.  $\arccos x$  vs  $\cos x$

### §4.3 Monotonic Functions and the First Derivative Test

**Definition.** Let  $f$  be a function defined on an interval  $I$  and let  $x_1$  and  $x_2$  be two distinct points in  $I$ .

1. If  $f(x_2) > f(x_1)$  whenever  $x_1 < x_2$ , then  $f$  is said to be **increasing** on  $I$ .
2. If  $f(x_2) < f(x_1)$  whenever  $x_1 < x_2$ , then  $f$  is said to be **decreasing** on  $I$ .

Ex. (a)  $y = x^3$ ,  $x \in \mathbb{R}$

(b)  $y = \sqrt{x}$ ,  $x \geq 0$

**Corollary 3.** Suppose that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f'(x) > 0$  at each point  $x$  in  $(a, b)$ , then  $f$  is increasing on  $[a, b]$ . If  $f'(x) < 0$  at each point  $x$  in  $(a, b)$ , then  $f$  is decreasing on  $[a, b]$ .

**Example 1.** Find the critical points of  $f(x) = x^3 - 12x - 5$  and identify the open intervals on which  $f$  is increasing and on which  $f$  is decreasing.

*Solution.*  $f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2)$ . Solve  $3(x + 2)(x - 2) = 0 \Rightarrow x = -2$  and  $x = 2$  critical points. Making a table and using testing point method show that  $f' > 0$  on  $(-\infty, -2)$ , increasing;  $f' < 0$  on  $(-2, 2)$ , decreasing;  $f' > 0$  on  $(2, \infty)$ , increasing.  $\square$

**Example 2.** Find the critical points of  $f(x) = x^{1/3}(x - 4)$ . Identify the open intervals on which  $f$  is increasing and decreasing. Find the

function's local and absolute extreme values.

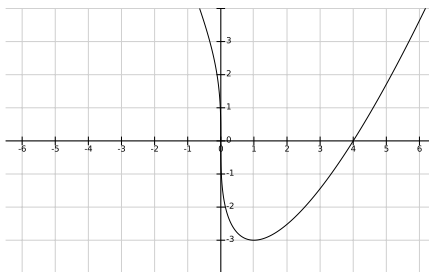
*Solution.*  $f'(x) = (x^{4/3} - 4x^{2/3})' = \frac{4}{3} \frac{x-1}{\sqrt[3]{x^2}}$ . Solve  $\frac{4}{3} \frac{x-1}{\sqrt[3]{x^2}} = 0$  to obtain  $x = 1$ , a critical point.

From the table we see  $f$  attains its local minimum  $f(1) = -3$  at  $x = 1$ .

TABLE 1.  $y = f(x) = x^{1/3}(x - 4)$

$x$	$-\infty < x < 1$	$1 < x < \infty$
$f'$	$< 0$	$> 0$
$f$	$\searrow$	$\nearrow$

The following figure plots the graph of the function  $y = f(x)$  over  $(-6.5, 6.5)$ .



□

**Example 3.** Find the critical points of  $f(x) = (x^2 - 3)e^x$ . Identify the open intervals on which  $f$  is increasing and decreasing. Find the function's local and absolute extreme values.

Ex. (a)  $y = x + \sin x$  (b)  $y = \frac{\sin x}{x}$  (c)  $y = e^{-x} \cos x$

#### §4.4 Concavity and Curve Sketching

**Definition.** The graph of a differentiable function  $y = f(x)$  is

- (a) **concave up** on an open interval  $I$  if  $f'$  is increasing on  $I$ ;
- (b) **concave down** on an open interval  $I$  if  $f'$  is decreasing on  $I$ .

**The Second Derivative Test for Concavity.** Let  $y = f(x)$  be twice-differentiable on an interval  $I$ . If  $f'' > 0$  on  $I$ , the graph of  $f$  over  $I$  is concave up. If  $f'' < 0$  on  $I$ , the graph of  $f$  over  $I$  is concave down.

**Example 1.** (a) The curve  $y = x^3$  is concave down on  $(-\infty, 0)$  and concave up on  $(0, \infty)$ .  
(b) The curve  $y = x^2$  is concave up on  $(-\infty, \infty)$ .

**Example 2.** Determine the concavity of  $y = 3 + \sin x$  on  $[0, 2\pi]$ .

**Definition.** A point  $(c, f(c))$  where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

**Example 3.** Determine the concavity and find the inflection points of the function  $f(x) = x^3 - 3x^2 + 2$ . See also Ex. in §4.1.

[Solution]  $f'(x) = 3x^2 - 6x = 3x(x - 2)$ . Solve  $f'(x) = 0$  and we obtain  $x = 0, 2$  are critical points in  $\mathbb{R}$ .  
 $f''(x) = 6x - 6 = 6(x - 1)$ . Solve  $6(x - 1) = 0$  we obtain  $x = 1$  is a reflection point.

TABLE 2.  $f(x) = x^3 - 3x^2 + 2$

Function \ $x$	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
$f'$	+	−	−	+
$f$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
$f''$	−	−	+	+
$f$	concave down	concave down	concave up	concave up

The function  $y = f(x)$  attains its local maximum  $y_{max} = f(0) = 2$  at  $x = 0$  and local minimum  $y_{min} = f(2) = -2$  at  $x = 2$ ; and has a reflection point at  $(1, 0)$  as Figure 2 shows. Note that  $f''(0) = -6 < 0$  (local max), and  $f''(2) = 6 > 0$  (local min.). See Figure 2.

**Example 4.** Determine the concavity and find the inflection points of  $f(x) = x^{5/3}$ .

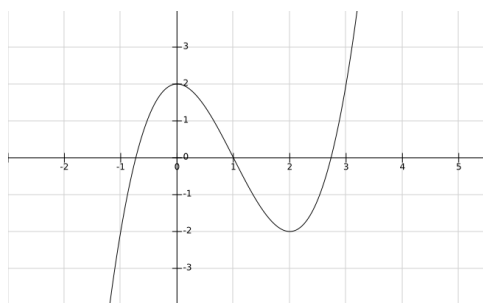


FIGURE 2. Plot of the cubic function  $f(x) = x^3 - 3x^2 + 2$  in Ex.3

**Theorem 5 (Second Derivative Test for Local Extrema).** Suppose  $f''$  is continuous on an open interval that contains  $x = c$ .

1. If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $x = c$ .
2. If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $x = c$ .
3. If  $f'(c) = 0$  and  $f''(c) = 0$ , then the test fails. The function  $f$  may have a local maximum, local minimum, or neither.

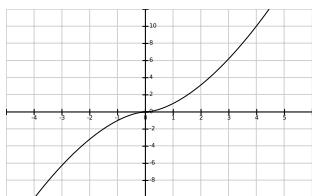
**Example 8.** Sketch a graph of the function  $f(x) = x^4 - 4x^3 + 10$ .

[Solution]  $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) = 0 \Rightarrow x = 0, 3$  critical points.

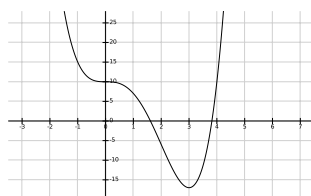
$f''(x) = 12x^2 - 24x = 12x(x - 2) \Rightarrow f''$  has zero at  $x = 0, 2$ .

TABLE 3.  $f(x) = x^4 - 4x^3 + 10$

$x$	$(-\infty, 0)$	$x = 0$	$(0, 2)$	$x = 2$	$(2, 3)$	$x = 3$	$(3, \infty)$
$f'$	$-$	0	$-$	$-$	$-$	0	$+$
$f$	$\searrow$	10	$\searrow$	-6	$\searrow$	-17	$\nearrow$
$f''$	$+$	0	$-$	0	$+$	$+$	$+$
$f$	$\cup$	infl. pt.	$\cap$	infl. pt.	$\cup$	local minimum	$\cup$



(A) Inflection point at  $(0, 0)$  of  $x^{5/3}$



(B) Inflection point at  $(0, 10)$  of  $x^4 - 4x^3 + 10$

FIGURE 3. Graphs for Ex.4 and Ex.8