§4.5 Indeterminate Forms and L'Hôpital's Rule

Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞.
These are the forms one cannot determine its limit simply by directing applying quotient/product rule etc.

Theorem 6 (L'Hôpital's Rule). Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, assuming that the limit on the right side of this equation exists.

Example 1. Find the limits: (a) $\lim_{x\to 0} \frac{3x - \sin x}{x}$ (b) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ (c) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

Solution. (a)

$$\lim_{x\to 0} \frac{3x - \sin x}{x} = \lim_{x\to 0} \frac{3 - \cos x}{1} = \frac{3-1}{1} = 2.$$

(b)

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-1/2}}{1} = \frac{1}{2}.$$

□

Example 2. Be careful to apply L'Hôpital's Rule correctly: $\lim_{x\to 0} \frac{1-\cos x}{x + x^2}$.

Example 3. Find (b) $\lim_{x\to 0^-} \frac{\sin x}{x^2}$.

Solution. Apparently this is of type $\frac{0}{0}$. Applying L'hopital Rule we obtain

$$\lim_{x\to 0^-} \frac{\sin x}{x^2} = \lim_{x\to 0^-} \frac{\cos x}{2x} = \frac{1}{0^-} = -\infty.$$

□

Example 4. Find the limits of these ∞/∞ forms: (a) $\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x}$ (c) $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Example 5. Find the limits of these $\infty \cdot 0$ form: (b) $\lim_{x\to 0^+} \sqrt{x}\ln x$.

Example 6. Find the limit of this $\infty - \infty$ form: $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution.

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}\quad\text{(by L.H. rule)}$$

$$\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\sin x}{\cos x+\cos x-x\sin x}\quad\text{(by L.H. rule)}$$

$$=\lim_{x\to 0}\frac{0}{2-0}=0.$$

□

Example 7. Apply L'Hôpital's Rule to show that $\lim_{x\to 0^+}(1+x)^{1/x}=e$.

Solution. Let $y=(1+x)^{1/x}$. Then $\ln y=\frac{1}{x}\ln(1+x)=\frac{\ln(1+x)}{x}$. This is of the form 0/0. We apply L'hopital rule to obtain

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{\frac{1}{1+x}}{1}\quad\text{(by L.H. rule)}$$

$$=\lim_{x\to 0}\frac{1}{1+x}=1.$$

Now we see that

$$\lim_{x\to 0}y=\lim_{x\to 0}e^{\ln y}=e^1=e.$$

□

Example 8. Find $\lim_{x\to\infty}x^{1/x}$.

§4.6 Applied Optimization*

Example 1. An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Exercise 8. A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

§4.8 Antiderivatives

Definition. A function F is an **antiderivative** of f on an interval I if $F'(x)=f(x)$ for all x in I.

Example 1. Find an antiderivative for: (a) $f(x)=2x$ (b) $g(x)=\cos x$.

Answer. (a) x^2+C; (b) $\sin x+C$ □

Theorem 8. If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

Example 2. Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Power Rule: The anti-derivative of x^n equals to $\frac{x^{n+1}}{n+1} + C$ if $n \neq -1$.

Exponential Rule: The anti-derivative of e^x equals to $e^x + C$.

Example 3. Find the general antiderivatives:

(a) $f(x) = x^5$ (b) $g(x) = \dfrac{1}{\sqrt{x}}$ (c) $h(x) = \sin(2x)$ (e) $j(x) = e^{-3x}$

Solution. (a) If $f(x) = x^5$, then the general anti-derivative is $F(x) = \frac{x^6}{6} + C$.
(j) We know the anti-derivative of e^x is given by $e^x + C$. What about the anti-derivative of e^{kx} with k being a constant? Since the derivative of e^{kx} is ke^{kx}, we see that for the anti-derivative, we need to divide it by a constant k. Hence with $k = -3$

$$j(x) = e^{-3x}$$
$$\Rightarrow J(x) = \frac{1}{-3}e^{-3x} + C = -\frac{1}{3}e^{-3x} + C.$$

□

Antiderivative Linearity Rules.

	Function	General antiderivative
1. *Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, k a constant
2. *Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Definition. The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by $\displaystyle\int f(x)dx$. The symbol $\displaystyle\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Example 6. Evaluate $\displaystyle\int (x^2 - 2x + 5)dx$.

Exercise 88. Right or wrong? $\displaystyle\int \frac{x\cos(x^2) - \sin(x^2)}{x^2}dx = \frac{\sin(x^2)}{x} + C.$

Answer. Wrong, because

$$\frac{d}{dx}\left(\frac{\sin(x^2)}{x} + C\right) = \frac{2x^2\cos(x^2) - \sin(x^2)}{x^2} \neq \frac{x\cos(x^2) - \sin(x^2)}{x^2}.$$

You can also check the indefinite integral of the function in online integral calculator □

MML

Video on Anti-differentiation and Indenite Integrals (29 minutes) including indefinite integrals and the power rule for anti-differentiation.