§5.2 Sigma Notation and Limits of Finite Sums

Example 1. Find the sums: (a) $\sum_{k=1}^{5} k$, (b) $\sum_{k=1}^{3}(-1)^k k$, (d) $\sum_{k=4}^{5} \frac{k^2}{k-1}$.

Algebra Rules for Finite Sums.

1 & 2. *Sum & Difference Rule:* $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

3. *Constant Multiple Rule:* $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)

4. *Constant Value Rule:* $\sum_{k=1}^{n} c = n \cdot c$ (Any number c)

Example 3. Find (d) $\sum_{k=1}^{n} \frac{1}{n}$.

Example 4. Show that the sums of the first n integers is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Example 5. Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity.

Definition. Let f be a bounded function defined on a closed interval $[a, b]$. Choose $n - 1$ points $\{x_1, x_2, \cdots, x_{n-1}\}$ between a and b that are in increasing order, and set $x_0 = a$ and $x_n = b$:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set of all of these points, $P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$, is called a **partition** of $[a, b]$. The partition P divides $[a, b]$ into the n closed subintervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$. The width of the kth subinterval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$ (here k is an integer between 1 and n). Choose a point in each subinterval $[x_{k-1}, x_k]$, called c_k, and form the product $f(c_k) \cdot \Delta x_k$. Finally we sum all these products to get $S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$. The sum S_P is called a **Riemann sum for f on the interval $[a, b]$**.

If we choose n subintervals all having equal width $\Delta x = (b - a)/n$ to partition $[a, b]$ and choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum, then it leads to the Riemann sum formula $S_n = \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right)$.

Definition. We define the **norm** of a partition P, written $\|P\|$, to be the largest of all the subinterval widths.

Example 6. Find the norm of the partition $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ of $[0, 2]$.

§5.3 The Definite Integral

Definition. Let f be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** if $J = \lim_{\|P\|\to 0} S_P = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$, no matter what choices of a partition P and points c_k are made. If the definite integral exists, then instead of writing J we write $\int_a^b f(x)dx$, and we say that f is **integrable** over $[a, b]$.

Theorem 1. If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

Theorem 2. When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules listed below.

1. *Order of Integration:* $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. *Zero Width Interval:* $\int_a^a f(x)dx = 0$
3. *Constant Multiple:* $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (Any constant k)
4. *Sum and Difference Rule:* $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. *Additivity:* $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Example 2. Suppose that $\int_{-1}^1 f(x)dx = 5$, $\int_1^4 f(x)dx = -2$, and $\int_{-1}^1 h(x)dx = 7$. Find

(a) $\int_4^1 f(x)dx$ (b) $\int_{-1}^1 [2f(x) + 3h(x)]dx$ (c) $\int_{-1}^4 f(x)dx$.

Definition. If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b, $A = \int_a^b f(x)dx$.

Example 4. Compute $\int_0^b xdx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

Ex. from MML

Video on the Definite Integral (28 minutes)