§*5.3 The Definite Integral* (Continued)

Exercise 20. Evaluate $\int_{-1}^{1} (1 - |x|)\, dx$.

Solution. Method I. The definite integral is equal to the area of under the graph of the "hat-function" $y = 1 - |x|$. So, we only need to find the area of the triangle $= \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$.

Method II. Note that $|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0. \end{cases}$ We split the integral into two parts:

$$\int_{-1}^{1} (1 - |x|)\, dx = \int_{-1}^{0} (1 - |x|)\, dx + \int_{0}^{1} (1 - |x|)\, dx$$
$$= \int_{-1}^{0} (1 - (-x))\, dx + \int_{0}^{1} (1 - x)\, dx$$
$$= (x + \frac{x^2}{2})\Big|_{-1}^{0} + (x - \frac{x^2}{2})\Big|_{0}^{1} = -(-\frac{1}{2}) + \frac{1}{2} = 1.$$

□

Definition. If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, which is also called its **mean**, is $\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)dx$.

Example 5. Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution. The integral of $f(x)$ over $[-2, 2]$ is equal to the area of a semi-circle of radius 2. Hence we have $\int_{-2}^{2} \sqrt{4 - x^2}dx = \frac{1}{2}\pi R^2 = \frac{1}{2}\pi(2)^2 = 2\pi$. By the definition,

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2-(-2)}(2\pi) = \frac{\pi}{2}.$$

□

Video The Integral (28 min) Definition of the integral. Signed area. Interval additivity property.

§*5.4 The Fundamental Theorem of Calculus*

Theorem 3 (The Mean Value Theorem for Definite Integrals). If f is continuous on $[a, b]$, then at some point c in $[a, b]$, $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$.

Theorem 4 (The Fundamental Theorem of Calculus, Part 1). If f is continuous on $[a,b]$, then $F(x) = \displaystyle\int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$: $F'(x) = \dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)$.

Example 2. Find dy/dx if

(a) $y = \displaystyle\int_a^x (t^3+1)dt$ (b) $y = \displaystyle\int_x^5 3t\sin t dt$ (c) $y = \displaystyle\int_1^{x^2} \cos t dt$ (d) $y = \displaystyle\int_{1+3x^2}^4 \frac{1}{2+e^t}dt$.

Solution. (a) The F.T.C. part I says that the derivative of $F(x) = \int_a^x f(t)dt$ w.r.t. x is equal to the integrand function evaluated at x, namely, $F'(x) = f(x)$. Thus

$$\frac{dy}{dx} = x^3 + 1.$$

(b) $y = -\int_5^x 3t\sin t dt \Rightarrow \frac{dy}{dx} = -3x\sin x$.
(c) Write the function y as a composite function $y = \int_1^u \cos t dt$, with $u = x^2$. By chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x) = 2x\cos x^2.$$

(d) Chain rule yields $y'(x) = \frac{dy}{dx} = -6x\frac{1}{2+e^{1+3x^2}} = -\frac{6x}{2+e^{3x^2+1}}$. □

Exercise 80. Find $f(4)$ if $\displaystyle\int_0^x f(t)dt = x\cos(\pi x)$.

Solution. F.T.C $\Rightarrow$

$$\frac{dy}{dx}\left(\int_0^x f(t)dt\right) = f(x).$$

On the other hand $\frac{dy}{dx}(x\cos(\pi x)) = \cos(\pi x) - \pi x\sin(\pi x)$. Hence

$$f(x) = \cos(\pi x) - \pi x\sin(\pi x)$$
$$\Rightarrow f(4) = \cos(4\pi) - \pi 4\sin(4\pi) = 1.$$

□

Theorem 4 Continued (The Fundamental Theorem of Calculus, Part 2). If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$.

Example 3. Calculate (a) $\displaystyle\int_{-\pi/4}^0 \sec x\tan x dx$ (c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx$ (d) $\displaystyle\int_0^1 \frac{dx}{x^2+1}$.

Solution. (a)

$$\int_{-\pi/4}^0 \sec x\tan x dx = \sec(x)\Big|_{-\pi/4}^0$$
$$= \sec(0) - \sec(-\pi/4) = \frac{1}{\cos 0} - \frac{1}{\cos(-\pi/4)} = 1 - \sqrt{2}.$$

(c)

$$\int_0^1 \frac{dx}{x^2+1} = \tan^{-1}(x)\Big|_0^1$$
$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}.$$

□

Theorem 5 (The Net Change Theorem). The net change in a differentiable function $F(x)$ over an interval $[a,b]$ is the integral of its rate of change: $F(b) - F(a) = \displaystyle\int_a^b F'(x)dx$.

Example 8. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution. Factorize $f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$. We see that $y = f(x)$ is positive over $[-1,0]$ and negative over $[0,2]$. So the area of the region between the x-axis and the graph of $f(x)$ is given by

$$A = \int_{-1}^2 |f(x)|dx = \int_{-1}^0 f(x)dx - \int_0^2 f(x)dx$$
$$= \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{-1}^0 - \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_0^2$$
$$= \frac{5}{12} - (-\frac{8}{3}) = \frac{37}{12}.$$

□

Video The Fundamental Theorem of Calculus (26 min) Average value theorem. The function $F(x) = \int_a^x f(s)\, ds$. The fundamental theorem of calculus.

§5.5 Indefinite Integrals and the Substitution Method

Example 1. Find the integral $\displaystyle\int (x^3+x)^5(3x^2+1)dx$.

Solution. Let $u = x^3 + x$, then $du = u'dx = (3x^2+1)dx$. We have

$$\int (x^3+x)^5(3x^2+1)dx = \int u^5 du = \frac{u^6}{6} + C = \frac{(x^3+x)^6}{6} + C.$$

□

Theorem 6 (The Substitution Rule). If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then $\displaystyle\int f(g(x)) \cdot g'(x)dx = \int f(u)du$.

Example 4. Find $\displaystyle\int \cos(7\theta + 3)d\theta$.

[Answer] $\frac{\sin(7\theta+3)}{7}+C$

Example 5. Find $\displaystyle\int x^2 e^{x^3}\,dx$.

[Answer] $\dfrac{e^{x^3}}{3}+C$

Example 6. Evaluate (1) $\displaystyle\int \sqrt{2x+1}\;dx$.

(2) $\displaystyle\int x\sqrt{2x+1}\;dx$

[Solution of (1)] $\quad \dfrac{(2x+1)^{3/2}}{3}+C.$

[Solution of (2)]. Let $u=2x+1$, then $x=\frac{u-1}{2}$, $dx=\frac{1}{2}du$.

$$\int x\sqrt{2x+1}dx = \int (\frac{u-1}{2})u^{1/2}\frac{du}{2}$$
$$=\frac{1}{4}\int (u-1)u^{1/2}du = \frac{1}{4}\left(\int u^{3/2}du - \int u^{1/2}du\right)$$
$$=\frac{1}{4}[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}+C] = \frac{1}{10}(2x+1)^{5/2}-\frac{1}{6}(2x+1)^{3/2}+C.$$

□

Integrals of the Tangent, Cotangent, Secant, and Cosecant Functions.

$\displaystyle\int \tan x dx = \ln|\sec x|+C$

$\displaystyle\int \cot x dx = \ln|\sin x|+C$

$\displaystyle\int \sec x dx = \ln|\sec x+\tan x|+C$

$\displaystyle\int \csc x dx = -\ln|\csc x+\cot x|+C$

Video Change of Variables (Substitution) (21 minutes)
Differentials. Using basic "u-substitutions" to find indefinite integrals and compute definite integrals.

§5.6 Definite Integral Substitutions and the Area Between Curves

Theorem 7 (Substitution in Definite Integrals). If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x)=u$, then $\displaystyle\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Example 1. Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.

Solution. Let $u = g(x) = x^3 + 1$, then $du = u'dx = 3x^2dx$. We have

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx = \int_{0}^{2} \sqrt{u}du$$
$$= \frac{2}{3}u^{3/2}|_0^2 = \frac{2}{3}(2^{3/2}) = \frac{4\sqrt{2}}{3}.$$

□

Example 2. Evaluate (a) $\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta d\theta$.

Solution. Let $u = \cot\theta$, then $du = u'd\theta = -\csc^2\theta d\theta$.

$$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta d\theta = -\int_{1}^{0} udu$$
$$= -\frac{u^2}{2}|_1^0 = 0 - (-\frac{1}{2}) = \frac{1}{2}.$$

□

Definition. If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b: $A = \int_a^b [f(x) - g(x)]dx$.

Example 4. Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Solution.

$$A = \int_0^1 (2e^{-x} + x) - \frac{e^x}{2}dx = -2e^{-x} + \frac{x^2}{2} - \frac{e^x}{2}|_0^1$$
$$= 3 - \frac{e}{2} - \frac{2}{e}.$$

□

Example 5. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution. First we found the intersection points of the parabola and the line. Solve

$$\begin{cases} y = 2 - x^2 \\ y = -x \end{cases} \quad \Rightarrow x_1 = -1, \quad x_2 = 2.$$

These will be the interval $[-1, 2]$. From the graph we see $f(x) \geq g(x)$ on $[-1, 2]$. By the definition of the area between two curves, we have

$$A = \int_a^b (f(x) - g(x))\, dx$$
$$= \int_{-1}^{2} \big(2 - x^2 - (-x)\big)\, dx$$
$$= 2x - \frac{x^3}{3} + \frac{x^2}{2}\Big|_{-1}^{2} = (4 - \frac{8}{3} + 2) - (-2 + \frac{1}{3} + \frac{1}{2}) = \frac{9}{2}\,.$$

□

Examples 6 & 7. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution. From the graph we see the area is divided into two parts: the parabola $\sqrt{x}$ over $[0, 2]$ and $[2, 4]$. To see this, first we found the intersection point(s) of the parabola and the line. Solve

$$\begin{cases} y = \sqrt{x} \\ y = x - 2 \end{cases} \qquad \Rightarrow x_1 = 1 \ (dropped), \quad x_2 = 4$$

where we observe that the point $(4, 2)$ is the only solution of the equations in the first quadrant.
The graph shows the area A is the sum of two parts

$$A = \int_0^2 \big(\sqrt{x} - 0\big)\, dx + \int_2^4 \big(\sqrt{x} - (x - 2)\big)\, dx$$
$$= \frac{2}{3}x^{3/2}\Big|_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right)\Big|_2^4$$
$$= \frac{2}{3}(2^{3/2}) + \big(\frac{2}{3}4^{3/2} - 8 + 8\big) - \big(\frac{2}{3}(2^{3/2}) - 2 + 4\big) = \frac{10}{3}\,.$$

□

Video Areas Between Curves (19 minutes)