Review Final Exam **Name**
Math 2160 **Id**

Read carefully each problem. Show all your work in order to justify and support your answer. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

(1) Which of the following equations are linear?
a) $xy + 3y = 1$
b) $3x - y + z = 9w$
c) $x \cos 15° + (2 - y) \sin 15° = \sqrt{2}$

(2) Write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve the matrix equation for $\mathbf{x} = [x_1, x_2]^T$ using Gauss elimination with back-substitution method.

$$\begin{cases} x_1 & +x_2 & = 15 \\ 2x_1 & +2x_2 & = 30 \end{cases}$$

(3) Find the inverse of the matrix (if it exists).
(a) $\begin{pmatrix} 10 & -5 \\ 5 & -3 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & -2 & -1 \\ 3 & -5 & -2 \\ 2 & -5 & -2 \end{pmatrix}$

(4) Compute the determinants.

$$(a) \quad \begin{vmatrix} 1 & 1 & 2 \\ 4 & 5 & 6 \\ 2 & 3 & -1 \end{vmatrix} \qquad (b^*) \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \lambda & 0 \\ 0 & \lambda & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

(5) Which of the following sets of vectors $x = [x_1, x_2, x_3, x_4]^T$ are subspace of $\mathbf{R}^4$?
a) All x such that $x_1 + x_2 = 7x_3$
b) All x such that $x_3 = 0$
c) All x such that $x_1 + x_4 = -12$
d) All x such that each x_i component is positive, that is, the first "I-quadrant" set $= \{x_i \geq 0, i = 1, 2, 3, 4\}$.

(6) Which of the following vectors, if any, is in the null space of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$?

a) $[-1\ 0\ 1\ 0]^T$ b) $[0\ 2\ 1\ -1]^T$ c) $[0\ 4\ 2\ -2]^T$

(7) [Testing for liner independence] Determine whether the following set S of vectors is linearly independent or linearly dependent?

(a) $S = \{(-2, 2), (3, 5)\}$ in $\mathbf{R}^2$ (4.4, #29)

(b) $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$ in $\mathbf{R}^3$ (4.4, #37)

(c) $S = \{9, x^2, x^2 + 1\}$ in P_2.

(8) [True or False]

(a) A set of vectors $S = \{v_1, v_2, \ldots, v_k\}$ in a vector space is called linearly dependent when the vector equation $c_1 v_2 + c_2 v_2 + \cdots + c_k v_k = 0$ has only the trivial solution.
(4.4, #59 (a))

(b) The set $S = \{(1, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ spans $\mathbf{R}^4$.

(c) A set $S = \{v_1, v_2, \ldots, v_k\}$, $k \geq 2$, is linearly independent if and only if at least one of the vectors v_j can be written as a linear combination of the other vectors.
(4.4, #60 (a))

(d) If a subset S spans a vector space V, then every vector in V can be written as a linear combination of the vectors in S.

(9) Determine which of the following statements are equivalent to the fact that a matrix A of size $n \times n$ is invertible?

a) A is nonsingular
b) The row space of A has dimension n
c) The determinant of A is nonzero
d) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any given $\mathbf{b}$ in $\mathbf{R}^n$
e) The system $A\mathbf{x} = \mathbf{0}$ has nonzero solution
f) The dimension of the null space of A is zero
g) The rows of A are linear independent
h) The columns of A are linear independent
i) The rank of A is n
j) A is row-equivalent to an identity matrix
k) All eigenvalues of A are nonzero
l) A has n linear independent eigenvectors
m) A is similar to an diagonal matrix
n) A can be written as the product of elementary matrices.

(10) Find all the eigenvalues and eigenvectors of the given matrix.

a) $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(11) [optional*] The matrix $A = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}$ has eigenvalues 5 and 8.

a) Find the eigenspaces E_5 and E_8 by solving $(\lambda I - A)\mathbf{x} = \mathbf{0}$.
b*) By the theorem in Section 7.3 we know that a symmetric matrix of size n by n is always diagonalizable, equivalently speaking, always has n linear independent eigenvectors. Find an ordered basis consisting of eigenvectors of A.
c*) Specify the matrices P and D in the diagonalization $P^{-1}AP = D$
d*) Find an orthogonal matrix U such that $U^{-1}AU = D$ (Hint: An (real) orthogonal matrix means $U^{-1} = U^T$ or equivalently $U^TU = UU^T = I_n$).

Solutions

(2) $x_1 = 15 - t$, $x_2 = t$; or equivalently, $\mathbf{x} = \begin{bmatrix} -t + 15 \\ t \end{bmatrix}$

(3) (b) Form the matrix $[A|I_3] = \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 3 & -5 & -2 & 0 & 1 & 0 \\ 2 & -5 & -2 & 0 & 0 & 1 \end{array}\right)$ Then use row operation to reduce to $[I_3|B]$. Hence the inverse equals $B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \\ -5 & 1 & 1 \end{pmatrix}$

(4) (a) -3 (b) $15(\lambda^2 - 1)$

(5) (a), (b) are vector subspaces; (because both of these sets of vectors contain a zero vector and also pass the addition and scalar multiplication test)
(c) is not a subspace (because it does not contain a zero vector);
(d) is not a subspaces (because it does not pass the scalar multiplication test: e.g., take $\mathbf{x} = (1, 2, 4, 6)$ in the set $\Sigma_+ := \{(x_1, x_2, x_3, x_4) : x_j \geq 0 \text{ for all } j\}$, then $(-1)\,\mathbf{x}$ is not in the set Σ_+)

(6) (b), (c)

(8) (a) False. (b) True. (c) False (d) True.

(9) (a), (b), (c), (d), (f), (g), (h), (i), (j), (k), (n)

(10) (a)

Solution. The eigenvalues are repeated, $\lambda_1 = \lambda_2 = 1$.

To solve for the corresponding eigenvectors, we plug $\lambda = 1$ in the linear equation $(I - A)X = 0$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We then obtain the eigenvector $X = (x_1, x_2) = (t, 0) = t(1, 0)$, t is any real number. There we find that only one linearly independent vector in $E_1 = span\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$. □

(b) $|\lambda I - A| = (\lambda + 1)^2(\lambda - 2)$

(11) (a)&(b) $E_5 = \text{span}\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$, $E_8 = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$

(c*) $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

(d*) The orthogonal matrix U consists of three eigenvectors that are orthogonal in $\mathbf{R}^3$. So we need to orthogonalise the base $u = [-1, 1, 0]^T$, $v = [-1, 0, 1]^T$, $w = [1, 1, 1]$. Since the third vector in E_8 is orthogonal the any vectors in E_5. We only need to orthogonalise the two vectors in E_5 by Grant-Schmidt method. Let

$$u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{v} = v - \frac{\langle v, u \rangle}{\langle u, u \rangle} u = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix},$$

Normalise the three vectors $u, \tilde{v}, w$ to obtain

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$