§*1.2 Gaussian Elimination and Gauss-Jordan Elimination* (Continued)

Row-Echelon Form and Reduced Row-Echelon Form. A matrix in **row-echelon form** has the properties below.
1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

Ex. Solve the system using either Gauss elimination or Gauss-Jordan elimination with back-substitution
First rewrite the system in the form of (associated) augmented matrix. Then perform Gaussian elimination.

$$-x + 2y = 1.5$$
$$2x - 4y = -3$$

[Answer: $x = 2t - 1.5$, $y = t$, $t \in \mathbb{R} = (-\infty, \infty)$.]

Ex. Solve the linear system

$$-2x_1 + 5x_2 = 10$$
$$x_1 + x_3 = 0$$
$$2x_1 - 3x_2 - x_3 = 5$$

Solution. Write the equations in the matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} -2 & 5 & 0 \\ 1 & 0 & 1 \\ 2 & -3 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}.$$

Then using the associated augmented matrix [A **b**] and Gaussian elimination we perform elementary row operations to obtain $x_1 = \frac{55}{9}$, $x_2 = \frac{40}{9}$, $x_3 = -\frac{55}{9}$. □

Example 5. Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

Answer. $(x_1, x_2, x_3, x_4) = (-1, 2, 1, 3)$ □

Example 6. Solve the system.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

Example 7. Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Answer. $(x, y, z) = (1, -1, 2)$. □

Example 8. Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

Solution.

$$\begin{aligned} [Ab] = \begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \end{aligned}$$

$\therefore \quad (x_1, x_2, x_3) = (-5t + 2, 3t - 1, t).$ □

Definition. Systems of linear equations in which each of the constant terms is zero are called **homogeneous**. A homogeneous system of m equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Theorem 1.1. Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.

Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 0 \end{aligned}$$

Solution. $(x_1, x_2, x_3) = (-5t, 3t, t) = t(-5, 3, 1)$. □

§2.1 Operations with Matrices

Equality of Matrices. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal when they have the same size $(m \times n)$ and $a_{ij} = b_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Matrix Addition. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their **sum** is the $m \times n$ matrix $A + B = [a_{ij} + b_{ij}]$.

Example 2. (a) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & -3 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & -8 \end{bmatrix}$

Scalar Multiplication. If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the scalar multiple of A by c is the $m \times n$ matrix $cA = [ca_{ij}]$.

Example 3. For the matrices A and B, find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Matrix Multiplication. If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** AB is an $m \times p$ matrix $AB = [c_{ij}]$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}.$$

Example 4. (a) Find the product AB, where $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

(b) Find the product A^2, where $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

[Answer:] (b) $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 6. Solve the matrix equation $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution. Use Gauss-Jordan form and back-substitution method. Row reduction

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -\frac{4}{7} \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{7} \\ 0 & 1 & -\frac{4}{7} \end{bmatrix}$$

$\therefore \quad \mathbf{x} = (x_1, x_2, x_3) = (\frac{1}{7}t, \frac{4}{7}t, t) = t(\frac{1}{7}, \frac{4}{7}, 1),\ t \in \mathbb{R}.$ $\square$