Math 2160 (Elementary Linear Algebra) Instructor: Dr. S. Zheng

§2.3 The Inverse of a Matrix

Definition. An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that $AB = BA = I_n$, where I_n is the identity matrix of order n. The matrix B is the (multiplicative) **inverse** of A. A matrix that does not have an inverse is **noninvertible** (or **singular**).

Theorem 2.7. If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

Example 3. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$.

Example 4. Show that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$ has no inverse.

Theorem 2.8. If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1}, A^k, cA, and A^T are invertible and the statements below are true.
1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1}\cdots A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$

Theorem 2.9. If A and B are invertible matrices of order n, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 2.10. If C is an invertible matrix, then the properties below are true.
1. If $AC = BC$, then $A = B$.
2. If $CA = CB$, then $A = B$.

Theorem 2.11. If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 8. Use an inverse matrix to solve the system

$$\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned}$$

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Definition. An $n \times n$ matrix is an **elementary matrix** when it can be obtained from the identity matrix I_n by a single elementary row operation.

Example 1. Which of the matrices below are elementary? For those that are, describe the corresponding elementary row operation.

$$\text{(a)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(b)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{(c)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{(d)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{(e)}\ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad \text{(f)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Theorem 2.12. Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is the product EA.

Example 3. Find a sequence of elementary matrices that can be used to write the matrix $A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$ in row-echelon form.

Solution.

$$A \longrightarrow \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$\therefore$ The elementary matrix corresponding to the row operation above are given in order as $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, meaning that

$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

□

Definition. Let A and B be $m \times n$ matrices. Matrix B is **row-equivalent** to A when there exists a finite number of elementary matrices $E_1, E_2, \cdots, E_k$ such that $B = E_k E_{k-1} \cdots E_2 E_1 A$.

Theorem 2.13. If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

Theorem 2.14. A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Example 4. Find a sequence of elementary matrices whose product is the nonsingular matrix $A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$.

[Clue: Row reduce A to I_2 through a sequence of elementary row operations $E_1, E_2, \ldots, E_k$.

$$A \xrightarrow{E_1} \cdots \xrightarrow{E_k} I_2.$$

That is, $E_k \cdots E_2 E_1 A = I \Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$, where each E_i^{-1} is an elementary matrix.]

Answer to Ex.4. : †According to this approach given in the Clue, we proceed as follows. The row operation $A := A_1 \to A_2 \to A_3 \to A_4 \to A_5 = I$ is represented by

$$\begin{aligned} E_1 A &= A_2 \\ E_2 A_2 &= A_3 \\ E_3 A_3 &= A_4 \\ E_4 A_4 &= I \end{aligned}$$

where

$$E_1 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$\Rightarrow$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_4^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$\therefore$

$$A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

□

Definition. If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an ***LU*-factorization** of A.

Example 6*. Find an LU-factorization of the matrix $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$.

Example 7. Solve the linear system.

$$\begin{aligned} x_1 - 3x_2 &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

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