§3.1 The Determinant of a Matrix

Determinant of a 2 × 2 Matrix. The **determinant** of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

Example 1. Find the determinants of (a) $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$.

Minors and Cofactors of a Square Matrix. If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

Example 2. Find all minors and cofactors of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

Determinant of a Square Matrix. If A is a square matrix of order $n \geq 2$, then the determinant of A is the sum of the entries in the first row of A multiplied by their respective cofactors. That is, $\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$.

Example 3. Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

[Answer: $|A| = 14$]

Theorem 3.1. Let A be a square matrix of order n. Then the determinant of A is

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

or

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Example 4. Find the determinant of $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}$.

Theorem 3.2. If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is, $\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$.

Example 6. Find the determinant of the lower triangular matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$.

[Answer: $|A| = 2(-2)(1)(3) = -12$]
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†An alternative method which I I call diagonal-product method: Copy the 1^{st} and the 2^{nd} columns of A to form 4^{th} and 5^{th} columns. Then obtain the determinant of A by adding (or subtracting) the products of the six diagonals as shown in the following example.

Example 5.

$$\begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{vmatrix} = 0 + 16 + (-12) - (-4) - 0 - 6 = 2.$$

§3.2 *Determinants and Elementary Operations*

Example 1. Compare the determinant of the matrices A and B.

(a) $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & -8 \\ -2 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$

Theorem 3.3. Let A and B be square matrices.
1. When B is obtained from A by interchanging two rows of A, $\det(B) = -\det(A)$.
2. When B is obtained from A by adding a multiple of a row of A to another row of A, $\det(B) = \det(A)$.
3. When B is obtained from A by multiplying a row of A by a nonzero constant c, $\det(B) = c\,\det(A)$.

Example 2. Find the determinant of $A = \begin{bmatrix} 0 & -7 & 14 \\ 1 & 2 & -2 \\ 0 & 3 & -8 \end{bmatrix}$.

Example 3. Find the determinant of $A = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{bmatrix}$ using **elementary column operations**.

Theorem 3.4. If A is a square matrix and any one of the conditions below is true, then $\det(A) = 0$.
1. An entire row (or an entire column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

Ex. #19. Find the determinant of $A = \begin{bmatrix} 2 & 1 & -1 & 0 & 4 \\ 1 & 0 & 1 & 3 & 2 \\ 3 & 6 & 1 & -3 & 6 \\ 0 & 4 & 0 & 2 & 0 \\ -1 & 8 & 5 & 3 & -2 \end{bmatrix}$.

[Answer: $|A| = 0$]

Example 6. Find the determinant of $A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$.

[Answer: $|A| = -135$]

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Quiz 1 (§3.2) **Name**

Math 2160 **Id**

Read carefully the question and avoid simple mistakes. Show all your work in order to support and justify your answer. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

(1) Find the determinant of the matrix A using *elementary row operations.*

$$A = \begin{bmatrix} -2 & 5 & 0 \\ 1 & 0 & 1 \\ 2 & -3 & -1 \end{bmatrix}$$