In 3.2 we discussed using elementary row/column operations to evaluate the determinant of a matrix.

§3.3 Properties of Determinants

Theorem 3.5. If A and B are square matrices of order n, then $\det(AB) = \det(A)\det(B)$.

Theorem 3.6. If A is a square matrix of order n and c is a scalar, then the determinant of cA is $\det(cA) = c^n \det(A)$.

Theorem 3.7. A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$.

Example 3. Determine whether each matrix has an inverse.

$$\text{(a)} \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution. (a) $det(A) = 0$.
(b) $det(A) \neq 0$.

By the augmented matrix method $[A\ I_3]$, we have $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ -1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$$|A^{-1}| = \begin{vmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{3} \begin{vmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{12}.$$

□

Theorem 3.8. If A is an $n \times n$ invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Example 4. Find $|A^{-1}|$ for the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$.

Solution. $|A| = 4 \Rightarrow |A^{-1}| = \frac{1}{4}$.

Check: $(A\ I_3)$-method $\Rightarrow A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ 1 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$.

$$\begin{vmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ 1 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{3}{4} & 0 \\ 1 & -\frac{3}{2} & 1 \\ \frac{1}{2} & -\frac{1}{4} & 0 \end{vmatrix} = (-1)^{2+3} \begin{vmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{vmatrix} = (-1)\left(\frac{1}{8} - \frac{3}{8}\right) = \frac{1}{4}.$$

□

Equivalent Conditions for a Nonsingular Matrix. If A is an $n \times n$ matrix, then the statements below are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row equivalent to I_n.
5. A can be written as the product of elementary matrices.
6. $\det(A) \neq 0$.

Proof. We show (1) $\Leftrightarrow$ (2).
First, show (1) $\Rightarrow$ (2). If A is invertible, then A^{-1} exists. The equation $A\mathbf{x} = \mathbf{b}$ has one and only solution $\mathbf{x} = A^{-1}\mathbf{b}$.
Conversely, show (2) $\Rightarrow$ (1). Assume for any $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has a (unique) solution. We take $\mathbf{b} = \mathbf{0}$, then the system $A\mathbf{x} = \mathbf{0}$ has a (unique) solution, namely, the trivial solution. To show A is invertible, we row reduce A into an echelon form $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$. Then on the last row of $\tilde{A}$ we must have a leading $\tilde{a}_{nn} = 1$; otherwise there will be infinitely many solutions due to a free variable x_n.
Now since A is row equivalent to an upper diagonal matrix with leading 1's on the main diagonal. We see A is is row equivalent to an identity matrix. That is A, is invertible. $\square$

Example 5. Which of the systems has a unique solution?

$$\text{(a)} \quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

[Solution] (a) $\det(A) = 0$, (b) $\det(A) = -12 \Rightarrow$ the system (b) has an unique solution.

Theorem 3.9. If A is a square matrix, then $\det(A) = \det(A^T)$.

Proof. One can verify the theorem by considering some concrete example, say, $A = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 1 & 0 \\ 2 & 6 & 3 \end{bmatrix}$ $\square$

Appendix A. Proof of Theorem 3.8

Proof of Theorem 3.8. Elementary matrices are of the following three forms (2 by 2):

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \ (c \neq 0) \quad \textit{and} \quad E_3 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} c^{-1} & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}.$$

We see each E_i^{-1} has a determinant $\det(E_i^{-1}) = (\det(E_i))^{-1}$ for $i = 1, \ldots, q$.

If $A = E_1 \cdots E_q$ are product of elementary matrices, then $A^{-1} = E_q^{-1} \cdots E_1^{-1}$. Hence

$$\begin{aligned} \det(A^{-1}) &= \det(E_q^{-1}) \cdots \det(E_1^{-1}) \\ &= \det(E_q)^{-1} \cdots \det(E_1)^{-1} \\ &= \left(\det(E_q) \cdots \det(E_1)\right)^{-1} = \left(\det(A)\right)^{-1}. \end{aligned}$$

□

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