§4.2 *Vector Spaces* (Continued)

Example 3. Show that the set of all 2×3 matrices with the operations of matrix addition and scalar multiplication is a vector space.

Example 4. Let P_2 be the set of all polynomials of the form $p(x) = a_0 + a_1x + a_2x^2$, where a_0, a_1, and a_2 are real numbers. The sum of two polynomials $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ is defined in the usual way,

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

and the scalar multiple of $p(x)$ by the scalar c is defined by

$$cp(x) = ca_0 + ca_1x + ca_2x^2.$$

Show that P_2 is a vector space.

Example 5. Let $C(-\infty, \infty)$ be the set of all real-valued continuous functions defined on the entire real line. Addition is defined by $(f + g)(x) = f(x) + g(x)$. Scalar multiplication is defined by $(cf)(x) = c[f(x)]$. Show that $C(-\infty, \infty)$ is a vector space.

Theorem 4.4. Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the properties below are true.
1. $0\mathbf{v} = \mathbf{0}$ 2. $c\mathbf{0} = \mathbf{0}$ 3. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$. 4. $(-1)\mathbf{v} = -\mathbf{v}$.

Example 6. The set of all integers (with the standard operations) does not form a vector space.

Example 7. The set of all second-degree polynomials is not a vector space.

Example 8. Let $V = R^2$, the set of all ordered pairs of real numbers, with the standard operation of addition and the nonstandard definition of scalar multiplication $c(x_1, x_2) = (cx_1, 0)$. Show that V is not a vector space.

§4.3 *Subspaces of Vector Spaces*

Definition. A nonempty subset W of a vector space V is a **subspace** of V when W is a vector space under the operations of addition and scalar multiplication defined in V.

Example 1. Show that the set $W = \{(x_1, 0, x_3) : x_1 \text{ and } x_3 \text{ are real numbers}\}$ is a subspace of R^3 with the standard operations.

Theorem 4.5. If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the two closure conditions listed below hold.
1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.

2. If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

Example 2. Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$, with the standard operations of matrix addition and scalar multiplication.

Example 3. Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2,2}$ with the standard operations.

Example 4. Show that $W = \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2.

Example 6. Determine whether each subset is a subspace of R^2.
(a) The set of points on the line $x + 2y = 0$ (b) The set of points on the line $x + 2y = 1$

Example 8. Determine whether each subset is a subspace of R^3.
(a) $W = \{(x_1, x_2, 1) : x_1 \text{ and } x_2 \text{ are real numbers}\}$
(b) $W = \{(x_1, x_1 + x_3, x_3) : x_1 \text{ and } x_3 \text{ are real numbers}\}$

§4.4 Spanning Sets and Linear Independence

Definition. A vector $\mathbf{v}$ in a vector space V is a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V when $\mathbf{v}$ can be written in the form $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots c_k\mathbf{u}_k$ where $c_1, c_2, \cdots, c_k$ are scalars.

Example 1. (a) For the set of vectors $S = \{(1, 3, 1), (0, 1, 2), (1, 0, -5)\}$ in R^3, the first vector is a linear combination of the other two.

Example 2. Write the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of vectors in the set $S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$.

Example 3. If possible, write the vector $\mathbf{w} = (1, -2, 2)$ as a linear combination of vectors in the set S in Example 2.

Definition. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be a subset of a vector space V. The set S is a **spanning set** of V when every vector in V can be written as a linear combination of vectors in S. In such cases it is said that S **spans** V.

Example 5. Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3.

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