**Math 2160** (Elementary Linear Algebra) Dr. J. S. Zheng

Course web

A systematic way of synchronously organizing lecture notes by summarizing the text and connecting to the virtual assignment

§*7.2 Diagonalization* (Continued)

**Theorem 7.6.** If an $n \times n$ matrix $A$ has $n$ distinct eigenvalues, then the corresponding eigenvectors are linearly independent and $A$ is diagonalizable.

§*7.3* Diagonalization*

Definition. A matrix $A = (a_{ij})_{n\times n}$ is orthogonal if $A^T A = AA^T = I$. That is, the inverse of $A$ is the transpose of that matrix.

**Proposition 1.** *Let $A = [v_1, v_2, \ldots, v_n]$ be a matrix with $n$ columns $v_j$, $1 \le j \le n$ Then $A$ is orthogonal* $\Longleftrightarrow$

$$\langle v_j, v_k \rangle = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

Definition. [Orthogonal diagonalization] A square matrix $A$ is orthogonally diagonalizable provided there exists an orthogonal matrix $P$ such that

$$P^T AP = D \tag{1}$$

where $D$ is a diagonal matrix.

†If $A$ is orthogonally diagonalizable then $AP = PD$, which means the $j$-th column of $P$ is an eigenvector corresponding to the eigenvalue $\lambda_j$.

This example illustrate how to find the eigenvalues and orthogonal eigenvectors of a given matrix.

Example. Given matrix $A = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}$ has eigenvalues 5 and 8.

(a) Find the eigenspaces $E_5$ and $E_8$ by solving $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

(b) By the theorem in Section 7.3 we know that a symmetric matrix of size $n$ by $n$ is always diagonalizable, equivalently speaking, always has $n$ linear independent eigenvectors. Find an ordered basis consisting of eigenvectors of $A$.

(c) Specify the matrices $P$ and $D$ in the diagonalization
$P^{-1}AP = D$

(d) Find an orthogonal matrix $U$ such that $U^{-1}AU = D$. Recall an (real) orthogonal matrix means $U^{-1} = U^T$ or equivalently $U^TU = UU^T = I_n$).

[Solution] (a) and (b). $E_5 = \text{span}\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\}$, $E_8 = \text{span}\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}$

(c*) $P = \begin{pmatrix}-1 & -1 & 1\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix}$, $D = \begin{pmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 8\end{pmatrix}$

(d*) The orthogonal matrix $U$ consists of three eigenvectors that are orthogonal in $\mathbb{R}^3$. So we need to orthogonalise the base $u = [-1, 1, 0]^T$, $v = [-1, 0, 1]^T$, $w = [1, 1, 1]$. Since the third vector in $E_8$ is orthogonal the any vectors in $E_5$. We only need to orthogonalise the two vectors in $E_5$ by Grant-Schmidt method. Let

$$u = \begin{pmatrix}-1\\1\\0\end{pmatrix}, \quad \tilde{v} = v - \frac{\langle v, u\rangle}{\langle u, u\rangle}u = \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{pmatrix},$$

Normalise the three vectors $u, \tilde{v}, w$ to obtain

$$U = \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}$$

Cengage Sample assignment. WebAssign: List of all sections

Video on orthogonal diagonalization on a symmetric matrix

(1) Orthogonal diagonalization $2 \times 2$
(2) Orthogonal diagonalization $3 \times 3$

Previous videos reference from *math is power*

(1) Find the Eigenvalues and Corresponding Unit Eigenvectors of a $2 \times 2$ Matrix
(2) Find the Corresponding Eigenvectors Given an Eigenvalues (Complex)
(3) Find the Eigenvalues and Corresponding Unit Eigenvectors of a $3 \times 3$ Matrix