Review Test 2	**Name**
Math 2242	**Id**

Put your name and the question number on each page.
Put a box around the final answer to a question.
(use the back of the page if necessary).
You must Show your work in order to get credits or partial credits.

1. Evaluate the limit of the sequence and explain how you obtained your answer.
(i) $\lim_{n\to\infty} \frac{n+(-1)^n}{\ln n}$
(ii) $\lim_{n\to\infty}(1-\frac{1}{3n})^{2n}$
(iii)$\lim_{n\to\infty} \frac{2^{n-1}+5n}{3^{n+2}}$
(iv) $\lim_{n\to\infty} e^{-\sqrt{n}} \ln n$
v) $\lim_{n\to\infty} \frac{n!}{n^n}$.

2. Determine whether or not the series converges, and if so, find its sum.
(i) $\sum_{n=1}^{\infty} \sin(\frac{\pi}{2}-\frac{1}{n})$
(ii) $\sum_{n=0}^{\infty}(-0.33)^n$
(iii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$
(iv) $\sum_{n=2}^{\infty} n^{-2}\ln n$
(v) $\sum_{n=1}^{\infty}[1+(-1)^n]$

3. Evaluate the integrals.
a) $\int \frac{1}{x(\ln x)^{3/2}}\,dx$
b) $\int_0^1 \frac{1}{\sqrt{x^2+1}}dx$
c) $\int \sqrt{\frac{1-x}{1+x}}\,dx$ (hint: let $x = \cos 2u$ and recall $1+\cos 2u = 2\cos^2 u$ and $1-\cos 2u = 2\sin^2 u$)
d*) $\int x(\sqrt{x}+1)^{\frac{1}{3}}dx$

4. Evaluate the definite integrals:
a) $\int_0^2 \tan^{-1}(1-x)\,dx$

b) $\int_{-\pi}^{\pi} \frac{\sin x}{\sqrt{3-\sin^2 x}}\, dx$
d) $\int_{-\pi/8}^{\pi/8} \sec(2x)\tan^3(2x)dx$
e) $\int_{-\pi}^{\pi} \sin^2 x \cos^2 x dx$

5. Find the following limits:

$$(a) \quad \lim_{x\to 0} \frac{\sin^{-1} x}{x}$$

$$(b) \quad \lim_{x\to +\infty} x^2 e^{-(x+2)}$$

Fill in the blanks or parenthesis in Problems **6** to **9**.

6. **Integral Test:** $a_n > 0$. (a) Let $f : [1, \infty) \to \mathbf{R}$ be so that

- $a_n = f(n)$ for each $n \in \mathbf{N}$
- f is a ____________________________ function
- f is a ____________________________ function
- f is a ____________________________ function .

Then $\sum a_n$ converges if and only if ____________________________ converges.

(b) [Bonus] According to the integral test, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is comparable to the p-integral $\int_1^{\infty} \frac{dx}{x^p}$. Thus

- If $p >$ ________ then $\sum \frac{1}{n^p}$ converges.
- If $\rho <$ ________ then $\sum \frac{1}{n^p}$ diverges.
- If $p =$ ________ then $\sum \frac{1}{n^p}$ ________

7. (a) **Comparison Test:** $a_n > 0$

- If $0 \le a_n \le b_n$ for all $n \in \mathbf{N}$ and $\sum b_n$ ________________, then $\sum a_n$ ________________.

- If $0 \leq b_n \leq a_n$ for all $n \in \mathbf{N}$ and $\sum b_n$ __________________, then $\sum a_n$ __________________.

8. **Root Test:** $a_n > 0$. Let $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$

- If $\rho <$ _________ then $\sum a_n$ converges.
- If $\rho >$ _________ then $\sum a_n$ diverges.
- If $\rho =$ _________ then the test is inconclusive.

9. $n^{\mathbf{th}}$**-term test:** Let $\{a_n\}$ be an arbitrary sequence.

(a) If $\sum_n a_n$ converges, then $\lim a_n =$ _________
(b) If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ _________.

10. Use partial fractions to decompose the rational function $P(x)/Q(x)$ as is the integrand, then evaluate the integral.

a)

$$\int \frac{x^2+4x+4}{x^3+2x}dx$$

b)

$$\int \frac{7s+4}{(s-2)(s+4)} \quad ds$$

c)

$$\int_0^1 \frac{u^3}{(u+1)^2}du$$

d)

$$\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}dx$$

Hint: $\frac{P(x)}{Q(x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

e)

$$\int \frac{x}{x^4+4x^2+8}dx$$

Hint: complete square of the denominator $Q(x)$ first.

11. L'Hopital's Rule. Determine whether the limit exists, if so, find the limit.

$$a)\ \lim_{t\to 0} \frac{t-\sin t}{\tan t}$$

$$b)\ \lim_{y\to 2} \frac{y^2+6}{y-2}$$

$$c)\ \lim_{z\to 1} \frac{z^2+4z-5}{z^3-1}$$

$$d)\ \lim_{t\to \infty} \frac{\ln(t^2+5t)}{\ln t}$$

$$e)\ \lim_{r\to 0}(1+3r)^{\frac{1}{r}}$$

12. Find the value of the sum of the geometric series.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{5}\right)^n$$

Hint: If $|r| < 1$ and c is a constant, then

$$\sum_{n=1}^{\infty} cr^n = \frac{c\,r}{1-r}$$

13. Consider the series $\sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n = \frac{5^n}{n!}$. Use ratio test to answer the question

answer: $\dfrac{a_{n+1}}{a_n} =$

$$\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n!}$$

☐ absolutely convergent

☐ conditionally convergent

☐ divergent

(Hint: A series $\sum b_n$ is called absolutely convergent if $\sum |b_n|$ converges. It is called conditionally convergent if $\sum b_n$ converges but $\sum |b_n|$ diverges)

14. Does the following series converge or diverge? Explain your answer by stating which kind of test you are using and how it works.

(i) (10 points) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{(n^2-6n+9)^{0.6}}$

(ii) (10 points) $\displaystyle\sum_{n=2}^{\infty} ne^{-n^2}$

15*. Find the Taylor polynomial $p_4(x)$ of degree 4 for the functions
(i) $f(x) = \ln(1+x^2)$, $-1 \leq x \leq 1$
(ii) $g(x) = \sqrt{3+x}$, $-3 \leq x \leq 3$
(iii) Use the solutions to (i) and (ii) to approximate $\ln 2$ and $\sqrt{3.1}$ respectively.

16. Determine whether the improper integral converges. If it does, find the value of the integral.

$$a)[10] \quad \int_0^2 \ln x \, dx$$

$$b)[10] \quad \int_1^\infty \frac{1}{x^{0.99}} \, dx$$

$$(a)[5] \quad \int_0^1 \ln(1-x) \, dx$$

$$(b)[5] \quad \int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}} \, dx$$

17. Does the following series converge or diverge? Explain your answer by stating which kind of test you are using and how it works.

$$a)[10] \quad \sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$b)[10] \quad \sum_{n=0}^{\infty} (-1)^n \frac{1}{n^{0.6}}$$

18. Let b_n be a decreasing sequence of positive numbers with $lim_{n\to\infty} b_n = 0$. The alternating series test says that the series $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ converges to a finite number s. Futhermore if we define the jth truncation error as $E_j := s - S_j$, then

$$|E_j| < b_{j+1}.$$

(As usual S_j is the jth partial sum of the series, $S_j = \sum_{n=1}^{j}(-1)^{n-1} b_n$).

Base on the above fact, explain clearly why

$$|\sin(1) - (1 - \frac{1}{3!} + \frac{1}{5!})| < 1/5000.$$

(Hint: $E_3 := \sin(1) - p_3(1)$; the power series expansion is $\sin x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$ $\sin(1) \approx 0.841471$)

19. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$.
a)[10] Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$
b)[5] Compute $\frac{d}{dx} f(x)$ (*hint*: term by term differentiation)
c)[5] Using b) and the formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $(-1 < x < 1)$ to find a compact expression for $f(x)$, $-1 < x < 1$.

20* (bonus) It is known that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, -1 < x < 1$.
a)[8]Show that $\ln \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}, -1 < x < 1$.
b)[6]Using a), show that $\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}$.
c)[6] Using the fact that

$$\sum_{n=N}^{\infty} \frac{1}{n2^n} \leq \sum_{n=N}^{\infty} \frac{1}{N2^n}$$

estimate $\ln 2$ with an error less than 0.01.