Review Final Exam — Math 2242

Name

Id

Read each question carefully. Avoid making simple mistakes! Put a box around the final answer to a question (Use the back of the page if necessary). For full credit you must *show your work*. You must have enough written work, including explanations when called for, to justify your answers.

1. Let $f(x) = x^3 - 6x^2 + 9x$.
a) Find the largest open interval I containing 0 on which f has an inverse. b) Compute $(f^{-1})'(0)$. (Hint: $f(0) = 0$)

2. Determine if the following limit exist or not. Find the limit if it exists.

a) $\lim\limits_{x\to 0+} \dfrac{\tan^{-1}(2x)}{2x}$ (Hint: $(\tan^{-1} x)' = \frac{1}{x^2+1}$)

b) $\lim\limits_{x\to +\infty} (x^2 + 500)e^{-x^2}$

c) $\lim\limits_{n\to\infty} \dfrac{\cos(\sqrt{n})}{\sqrt{n}}$

d) $\lim\limits_{n\to\infty} \sin(\dfrac{\pi n}{2})$

e) $\lim\limits_{n\to\infty} \sqrt[n]{4n+7}$

3. Evaluate the integrals.
a) $\int \frac{1}{x(\ln x)^{3/2}}\, dx$
b) $\int_0^1 \frac{1}{\sqrt{x^2+1}} dx$ (Hint: Substitution $x = \tan t$. Note $1 + \tan^2 t = \sec^2 t$)
c) $\int x \ln x dx$

4. Fill in the blanks.

a. $\displaystyle\int e^{-3u}\,du \;=$ __________________________

b. $\int \cos(u+\pi)du \;=$ __________________________ $+C$

c. $\int \sin^2 u\,du \;=$ ______________________________ $+C$
(Hint: $\sin^2 u = \frac{1}{2}(1-\cos 2u)$)

d. If $a>0$ is a constant then

$\int \frac{u}{a^2+u^2}\,du \;=$ ______________________________ $+C$

e. If a is a constant and $a\neq 0$ then

$\int \frac{1}{a^2-u^2}\,du \;=$ ______________________________________ $+C$
(Hint: $\frac{1}{a^2-u^2} = \frac{1}{2a}(\frac{1}{a-u}+\frac{1}{a+u})$)

f. $\cos^{-1}(-\frac{\sqrt{3}}{2}) \;=$ _______________ (Hint: Range of inverse cosine is $[0,\pi]$)

g. $\tan(3\pi/4) =$_______________

h. $3^{\log_3 9} =$_______________

i. $\log_{10} 0.1 =$_______________

j. According to the integral test, the p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is comparable to the p-integral $\int_1^{\infty}\frac{dx}{x^p}$. Thus

- If $p>1$, then $\sum \frac{1}{n^p}$ ________________________.
- If $p\leq 1$, then $\sum \frac{1}{n^p}$ ________________________.

k. **Squeezing Theorem for Sequence** Let $a_n \leq c_n \leq b_n$. If $\lim_n a_n = \lim_n b_n = L$, L a finite number, then $\lim_n c_n$__________________________

l. **Root Test: Let** $a_n>0$. Let $\rho = \lim_{n\to\infty}\sqrt[n]{a_n}$

- If $\rho<1$, then $\sum a_n$ ________________________
- If $\rho>1$, then $\sum a_n$ ________________________.
- If $\rho=1$, then the test is ________________________.
- Given an example that illustrates root test__________________________________.

m. $n^{\mathbf{th}}$**-term test:** Let $\{a_n\}$ be an arbitrary sequence.
If $\sum_n a_n$ converges, then $\lim a_n = 0$. Give an example to explain the n-th term test: ______________________________

5. Determine whether the improper integral converges. If it does, find the value of the integral.

$$a) \quad \int_0^2 \ln x \, dx$$

$$b) \quad \int_1^\infty \frac{1}{(x-1)^{0.99}} \, dx$$

6. Does the following series converge or diverge? Explain your answer by stating which kind of test you are using and how it works.

$$a) \quad \sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$b) \quad \sum_{n=0}^{\infty} (-1)^n \frac{1}{n^{0.6}}$$

7. Find the Taylor polynomial $p_3(x)$ of degree 3 for the functions
a) $g(x) = \tan^{-1} x$, $-\infty < x < \infty$, then use it to give an approximation value for $\pi/4$
b) $h(x) = \sqrt{1+x^4}$, then use it to approximate $\sqrt{2}$.
How about $p_8(x)$, $p_{17}(x)$, $p_{21}(x)$?

8. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$.
a) Determine the radius of convergence of this power series.
b) Compute $\frac{d}{dx} f(x)$ (*hint*: term by term differentiation)
c) Using b) and the formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $(-1 < x < 1)$ to find a compact expression for $f(x)$, $-1 < x < 1$.
d) What is the value of $f(-1)$?

9. Find the Taylor polynomial $p_3(x)$ of degree 3 for the function $\sin x$, then use it to give an approximation value of $\sin(1)$. What is the absolute value of the error E_3 in your approximation? ($E_3 := \sin(1) - p_3(1)$; the power series expansion is $\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$ $\sin(1) \approx 0.841471$)

10. Let $a > 0$ be a constant. The parametric equation of an *astroid* is given by $x = a\cos^3 t$ and $y = a\sin^3 t$ for $0 \le t \le 2\pi$.
(1) What is the Cartesian equation for the curve?

(2) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(\frac{a}{8}, \frac{3\sqrt{3}a}{8})$.
(3) Give an integral expression of
a) The total arc length of the astroid
b) The area enclosed by the astroid curve.
c**) (optional) The area S of the surface generated by revolving the astroid around the x axis. Do *not* attempt to evaluate the integral (*hint*: S should be twice the surface area of the part for which $0 \le t \le \pi/2$)

Solutions

1. a) $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 2(x-1)(x-3)$. The zeros 1 and 3 divide the real axis into three parts: $(-\infty, 1)$, $(1, 3)$ and $(3, \infty)$. The first interval contains 0 where $f' > 0$ on $(-\infty, 1)$, which means f monotonically increases and hence f has an inverse on $(-\infty, 1)$.

b) By the inverse function derivative formula, we have

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Here $f(a) = 0$ and we need to find the value of a. Solve $x^3 - 6x^2 + 9x = 0$, or $x(x^2 - 6x + 9) = x(x-3)^2$ to get $x = 0$ or $x = 3$.

So $a = 0$ since only 0 belongs to the interval $(-\infty, 1)$ and 3 does not. Therefore we get

$$(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{9}.$$

2 (b) Let $u = x^2$, then

$$\lim_{x \to +\infty} \frac{x^2 + 500}{e^{x^2}} = \lim_{u \to +\infty} \frac{u + 500}{e^u}$$
$$\overset{\text{L'hopital}}{=} \lim_{u \to \infty} \frac{1}{e^u} = \frac{1}{\infty} = 0.$$

(c) Since $-1 \le \cos\sqrt{n} \le 1$, it follows that

$$\frac{-1}{\sqrt{n}} \le \frac{\cos(\sqrt{n})}{\sqrt{n}} \le \frac{1}{\sqrt{n}}.$$

Since $\lim_n \frac{1}{\sqrt{n}} = \lim_n -\frac{1}{\sqrt{n}} = 0$, we have by Squeezing theorem

$$\lim_n \frac{\cos(\sqrt{n})}{\sqrt{n}} = 0.$$

e)

$$\sqrt[n]{4n+7} = e^{\frac{1}{n}\ln(4n+7)}.$$

By L'hopital rule, $\lim_n \frac{1}{n}\ln(4n+7) = 0$. So

$$\lim_n \sqrt[n]{4n+7} = \lim_n e^{\frac{1}{n}\ln(4n+7)} = e^0 = 1.$$

7. a) From the able in the hand-out notes or from the Text 10.8 or 10.9, we know

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \qquad x \in [-1, 1]$$

So the Taylor polynomial of degree 3 is

$$p_3(x) = x - \frac{x^3}{3}$$

which approximates $\tan^{-1}(x)$. Substituting $x = 1$ we obtain

$$\tan^{-1}(1) \approx x - \frac{x^3}{3} = p_3(1) = 1 - 1/3 = 0.667$$

That is $\frac{\pi}{4} \approx 0.667$. (If you use more terms in the Talyor series p_8 or p_10, say, you will get better approximation value).

To find $p_8(x)$ (Taylor polynomial of degree 8), we use the first four terms to get

$$p_8(x) = \sum_{k=0}^{3} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7.$$

We see that $\tan^{-1}(1) \approx p_8(1) = 0.7238$ is a better approximation for $\pi/4$.

Similarly $p_{17}(1) \approx 0.81309$ and $p_{21}(1) \approx 0.76284$.

Second method. By the Taylor formula, if f has derivatives of all orders on an interval $(x_0 - \delta, x_0 + \delta)$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} x^k$$

is the Taylor series for f. Now $f(x) = \tan^{-1} x$, so we can compute the derivatives $f^{(k)}$ at $x_0 = 0$ for $k = 0, 1, 2, 3$ and we will get the same Taylor polynomial $p_3(x)$ as found above.

8. a) Let $a_n = x^n/n$. From

$$\frac{a_{n+1}}{a_n} = \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} = \frac{n}{n+1} x$$

we have $|a_{n+1}/a_n| \to |x|$. By Ratio Test, the series converges if $|x| < 1$.

So the radius of convergence is $R = 1$. Since $\sum_n (-1)^n \frac{1}{n}$ converges and $\sum_n \frac{1}{n}$ diverges, the interval of convergence is $[-1, 1)$.

b) and c) Term by term differentiation gives

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right)$$
$$= \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n}\right) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

for $-1 < x < 1$. Now integrating both sides of $f'(x) = \frac{1}{1-x}$ we obtain

$$f(x) - f(0) = \int_0^x f'(t)dt = \int_0^x \frac{1}{1-t} dt = -\ln(1-x).$$

Notice that $f(0) = \sum_1^{\infty} (0)^n/n = 0$. Thus we have

$$f(x) = -\ln(1-x), \quad -1 < x < 1.$$

10. (1) $x^{2/3} + y^{2/3} = a^{2/3}$

(2) $t = \pi/3$ corresponds to the point $(\frac{a}{8}, \frac{3\sqrt{3}a}{8})$.

$$\frac{dy}{dx}\Big|_{(\frac{a}{8}, \frac{3\sqrt{3}a}{8})} = \frac{dy/dt}{dx/dt}\Big|_{t=\pi/3} = \frac{3a\sin^2 t \cos t}{3a\cos^2 t(-\sin t)}\Big|_{t=\pi/3}$$
$$= -\frac{\sin t}{\cos t}\Big|_{t=\pi/3} = -\tan t|_{t=\pi/3} = -\sqrt{3}.$$

Let $y' = y'(x)$.

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx}$$
$$= \frac{dy'/dt}{dx/dt} = \frac{1}{3a\cos^4 t \sin t}$$

Hence

$$\frac{d^2y}{dx^2}\Big|_{(\frac{a}{8}, \frac{3\sqrt{3}a}{8})} = \frac{1}{3a\cos^4 t \sin t}\Big|_{t=\pi/3} = \frac{16\sqrt{2}}{3a}.$$

(3) a) Total arc length is equal to

$$L = 4\int_0^{\pi/2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

$$= 4\int_0^{\pi/2} \sqrt{(3a\cos^2 t(-\sin t))^2 + (3a\sin^2 t\cos t)^2} dt$$

$$= 4\int_0^{\pi/2} 3a\cos t\sin t dt = 12a\int_0^{\pi/2} \cos t\sin t dt.$$

b) The area enclosed by the astroid curve is equal to

$$A = 4\int_{x=0}^{x=a} |y| dx = 4\int_0^{\pi/2} |y(t)||x'(t)| dt$$

$$= 4\int_0^{\pi/2} |a\sin^3 t| 3a\cos^2 t(-\sin t) dt = 12a\int_0^{\pi/2} \sin^4 t\cos^2 t dt.$$

c**) The surface area, according to the formula in Section 11.2 for a parametric curve, is equal to

$$A = 2\int_{x=0}^{x=a} 2\pi|y|\sqrt{1 + y'(x)^2} dx = 4\pi\int_0^{\pi/2} |a\sin^3 t|\sqrt{x'(t)^2 + y'(t)^2} dt$$

$$= 4a\pi\int_0^{\pi/2} \sin^3 t(3a\sin t\cos t) dt = 12a^2\int_0^{\pi/2} \sin^4 t\cos t dt.$$