Review Test 1 **Name**
Math 2243 **Id**

Read each question carefully. Avoid simple mistakes. Put a box around the final answer to a question. (Use the back of the page if necessary).
You must show your work in order to get credits or partial credits.

1. Which of the points $P(6,2,0)$, $Q(-5,0,4)$, and $R(-1,-1,10)$ is closest to the plane $y=x$? Which point lies on the xy-plane?

2. Determine the center and radius of the sphere represented by the following equation.
$$x^2+y^2+z^2=4x-2y$$

3. Given two vectors: $\vec{a}=4\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\vec{b}=-\mathbf{i}-\mathbf{j}+2\mathbf{k}$, compute $\text{Proj}_b a$ and $\text{Comp}_b a$, that is, the projection of $\vec{a}$ onto the normal direction that is perpendicular to $\vec{b}$ (such that $\vec{a}=Proj_b a+Comp_b a$)

4. Evaluate $(\overline{a}+\overline{b})\bullet(2\overline{a}-3\overline{b})$ where $\overline{a}=<3,-2,-4>$ and $\overline{b}=<-1,0,1>$. Also, find $\overline{a}\times\overline{b}$.

5. a) Locate the point (x_0,y_0,z_0) at which the line intersects the given plane:
$$x=1-2t,\quad y=-3,\quad z=2-t;\quad x-y+2z+2=0$$
b) What is the direction angles of $\mathbf{v}=-2\mathbf{i}-\mathbf{k}$.

6. Find: a) A vector orthogonal to the plane containing the points: $P(1,0,-1)$, $Q(-2,1,1)$, and $R(4,-2,0)$. b) equation of the plane; and, c) the area of the triangle $\triangle PQR$.

7. Find a unit vector that is orthogonal to both $\mathbf{i}-\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$

8. (optional) Find the parametric equations for the line that passes through $(1,2,1)$ and is perpendicular to the line ℓ: $x=1+t$, $y=2$, $z=3-t$ with the assumption that these two lines are in the same plane.

9. Find an equation of the plane that passes through $A=(1,-1,2)$ and contains the line of intersection of the planes: $x+y-z=0$ and $y+2z-2=0$.

10. Prove the identity $\overline{a}\cdot(\overline{b}\times\overline{c})=-(\overline{a}\times\overline{c})\cdot\overline{b}$ by using the properties of cross product.

11. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t)=\mathbf{i}+\sin t\mathbf{j}-\sqrt{t}\mathbf{k}$ and $\mathbf{r}(1)=\mathbf{i}-\mathbf{j}$.

12. Locate the point of intersection between the two space curves: $\mathbf{r}_1(t) = < 2t, t-1, 1+2t^2 >$ and $\mathbf{r}_2(s) =< 2s, 1, (s+1)^2 >$. What is their angle of intersection?

13. Find the length of the curve $\mathbf{r}(t) = \sqrt{2}\, t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, $0 \le t \le \ln 2$.

14. Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t,

$$\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} - 8t\mathbf{k}.$$

Use your reparametrized space curve equations to find the point on the curve where the arc length is 2 ft from the point $t = 1$.

15. (Bonus) Compute a) the distance from the point $P(1, -1, 1)$ to the line $\frac{x}{2} = \frac{y-1}{3} = z$;

b) the distance from (0,0,1) to the plane $x + y + z = 1$.

c) Find the distance in $\mathbb{R}^3$ between the x-axis (the line $y = z = 0$) and the line

$$x - 3 = \frac{y-2}{2} = z + 1$$

d) How about the distance between the x-axis and the line $y = z = 6$

16. Compute a) the unit tangent $\mathbf{T}$, unit norm $\mathbf{N}$ (also called principal norm) and binormal $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, b) curvature κ, c) acceleration $\mathbf{a} = \mathbf{a_T} + \mathbf{a_N}$ of $\mathbf{r}(t) =< t, t^2, t^3 >$ at the point $(-1, 1, -1)$.

17. The graph in three dimensions of $\mathbf{r}(t)$ is a curve. The tangential scalar component of acceleration is $a_T = \frac{d}{dt}|\mathbf{v}|$ while the normal scalar component of acceleration is $a_N = \kappa|\mathbf{v}|^2$. Explain why you have to hold on when your car makes a sharp turn at high speed.

Part 2

1. Let $r(t) = (2t - 5)i + (2t^2 - 1)j$ in two dimensions. (a) Find the velocity and acceleration.

   (b) Graph this function, showing the orientation. Identify the graph. On your graph, show (and label) the position, velocity and acceleration vectors when $t = 1$.

2. Let $f(x, y) = \sqrt{81 - x^2 - y^2}$ (a) Find and show (graphically) the domain of this function.

   (b) On your graph, show the level curve corresponding to $z = 4$.

3. Let $g(x, y) = \frac{2}{x-y+5}$

   (a) Find and show (graphically) the domain of this function.

   (b) Find the equation of the level curve for $c = 3$ and show this level curve in the domain.

4. (a) Sketch and describe the graph of $r(t) = \langle 3\cos t, 3\sin t, 4t\rangle = (3\cos t)i + (3\sin t)j + 4tk$ in three dimensions. Tell me three specific points on the graph. Identify this graph by name.

   (b) Find T, the unit tangent vector.

   (c) Find N, the principal unit normal vector

   (d) Tell how to compute the curvature.

5. * Let $f(x,y) = 3x^4 \sin y + e^{5x}y^3 + x^3 + 2x - 6$. Find all first and second partial derivatives.

6. * A projectile is launched from height 25 feet, angle of thirty degrees, initial speed of 60 ft/sec. Thus we know that the path of the projectile is given by $r(t) = (15\sqrt{3}t)i + (12 + 30t - 16t^2)j$

   (a) When does the projectile impact the ground? How far downrange?

   (b) At what time does the projectile reach its maximum height?

7. Matching (place the most appropriate letter in each blank)

__	$\frac{x^2}{4} + \frac{y^2}{49} + \frac{z^2}{9} = 1$	A. Hyperboloid of two sheets
__	shaped like a Pringle	B. Ellipsoid
__	$\frac{x^2}{4} - \frac{y^2}{49} + \frac{z^2}{9} = 1$	C. Hyperboloid of one sheet
__	shaped like a nuclear cooling tower	D. Hyperbolic paraboloid
__	shaped like an egg	E. Elliptic paraboloid
__	$\frac{x^2}{4} - \frac{y^2}{49} - \frac{z^2}{9} = 1$	F. $u \cdot (v \times w) = 0$
__	volume of parallelepiped	G. $u \times v$
__	u and v are along the same line	H. $u /\!/ v$
__	twice as long as **u**, the opposite direction	I. $\lvert u \times v\rvert$
__	orthogonal to u and to v	J. $u \times v = 0$
__	u and v are orthogonal	K. $u \cdot v = 0$
__	area of parallelogram	L. $x + z = 5$
__	graph is a plane	M. *Cone*
____	$x^2 + y^2 - z^2 = 0$	N. $-2\mathbf{u}$

8. Fill in blanks:

   (a) $i \times j =$

   (b) $i \times i =$

   (c) $k \times j =$

Solution to Review Test 1 (The main credits come from your idea, work, steps and the answer that are correct)

1. $R = (-1, -1, 10)$ since R is on the plane $x = y$; $P = (6, 2, 0)$ since P is on the $xy-$plane.

2. Try complete the square to write the quadratic equation in an (equivalent) stand from

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

3. $\mathbf{Proj_b a} = (\mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|})\frac{\mathbf{b}}{|\mathbf{b}|}$ and by parallelogram rule (in a rectangle) $\mathbf{Comp_b a} = \mathbf{a} - \mathbf{Proj_b a}$.

4.

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} i & j & k \\ 3 & -2 & -4 \\ -1 & 0 & 1 \end{pmatrix}$$

5. a) Substitute $x = 1 - 2t, y = -3, z = 2 - t$; into $x - y + 2z + 2 = 0$ and solve for t.

b) Direction angles are $(\cos\alpha, \cos\beta, \cos\gamma)$, where α, β, γ are the angles between the vector $\mathbf{v}$ and x, y, z axis respectively.

$$\cos\alpha = \frac{-2}{\sqrt{5}}, \quad \cos\beta = \frac{0}{\sqrt{5}}, \quad \cos\gamma = \frac{-1}{\sqrt{5}}$$

6. a) $N = PQ \times PR$; (we can take the vector N to be the cross product of PQ and PR). $PQ = Q - P = \langle -3, 1, 2\rangle$ and $PR = R - P = \langle 3, -2, 1\rangle$. Use the 3 by 3 matrix to compute the determinant to obtain

$$N = PQ \times PR = \det \begin{bmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= 5i + 9j + 3k$$

b) Equation of the plance: $\mathbf{N} \cdot (x-1, y-0, z+1) = 0$, that is, $5(x-1) + 9y + 3(z+1) = 0$.

c) Area of $\triangle PQR = \frac{1}{2}\|PQ \times PR\| = \frac{1}{2}\sqrt{5^2 + 9^2 + 3^2} = \frac{\sqrt{115}}{2} = \frac{\sqrt{5 \cdot 23}}{2}$

7. Take the cross product.

8*. Obviously the desired line L should intersect the line ℓ. However it is quite computational if trying to find the intersection point. Instead, we try to find the normal line $\mathbf{N}$ of the plane Π that passes thru $A = (1, -1, 2)$ and ℓ first. This is easy because we can take the cross product of AB (B is any point on ℓ, say B=(1,2,3)) and $\mathbf{v} = (1, 0, -1)$.

Notice that $\mathbf{L}$ is perpendicular to both $\mathbf{N}$ and ℓ. So the direction vector for L can be taken as the cross product of N and $\mathbf{v}$. Use the point-$\mathbf{v}$ form and we are done.

Alternatively, we can first write the equation of the plane that passes thru $(1, 2, 1)$ and perpendicular to ℓ.

$$1 \cdot (x-1) + 0 \cdot (y-2) + (-1) \cdot (z-1) = 0$$

$$\text{simplify to get } x - z = 0$$

Substituting the parametric equation of ℓ into the equation of the plane $x-z=0$ we obtain $t=1$ and thus the point of intersection is $(2,2,2)$.

Using the other point $(1,2,1)$ we find the direction of L to be $\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} k$. Thus the equation of L is $\mathbf{r}(t) = \langle 1 + \frac{t}{\sqrt{2}}, 2, 1 + \frac{t}{\sqrt{2}} \rangle$

9. If we can find the normal line N of the plane, then the equation of the plane is given by the standard point-normal line form. But N can be taken as the product of the intersection line L and AB, where B is any point on L. The direction vector $\mathbf{v}$ of L can be obtained by taking the cross product of normals lines of the two intersecting planes. So the rest of the work is to compute by following the lines given above.

Second solution. We know point A is on the plane. We try to find two other points. we can take B=(-2,2,0) and C=(1,0,1) on the line of intersection. (because B, C satisfy both equations of the planes). Then find the cross product $N = AB \times AC$ which will be a normal line of the plane under consideration. The equation is then determined by point-normal form, say A and N, where $A = (1,-1,2)$, $AB = \langle -3, 3, 2 \rangle$, $AC = \langle 0, 1, -1 \rangle$

$$N = \det \begin{pmatrix} i & j & k \\ -3 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix} = -i - 3j - 3k$$

Point-normal form of the desired plane (using A and N):

$$-(x-1) - 3(y+1) - 3(z-2) = 0$$

10. Proof 1, the mixed product is the volume of the (3D)-parallelogram spanned by $\vec{a}, \vec{b}, \vec{c}$, the same can be said for the $\vec{c}, \vec{a}, \vec{b}$. The equation follows by noting that $\vec{c}, \vec{a} = -\vec{a} \times \vec{c}$.

Proof 2.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$(\vec{a} \times \vec{c}) \cdot \vec{b} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

From linear algebra or property of evaluating determinant of a matrix, switch two rows results in a negative sign for the determinant.

11. Integrating we get $\mathbf{r}(t) = \int \mathbf{r}'(t)dt = \int \mathbf{i} + \int \sin t\mathbf{j} - \int \sqrt{t}\mathbf{k} = t\mathbf{i} - \cos t\mathbf{j} - \frac{2}{3}t^{3/2}\mathbf{k} + \mathbf{C}$. Plugging in $t=1$ in $\mathbf{r}(t)$ above and use $\mathbf{r}(1) = \mathbf{i} - \mathbf{j}$ we can solve the constant vector $\mathbf{C}$.

12. We need to find the point (a,b,c) that lies on both curves, or, satisfies both parametric equations. To do so we write down the equations:

$$2t = 2s, t - 1 = 1, 1 + 2t^2 = (s+1)^2.$$

Then solve it to obtain t=2, s=2. The angle means the angle between two tangents or two velocities at the point (4,1,9), which can be computed using **dot product** (for the cosine).

In fact, the velocity of the first curve is $u(t) = dr_1/dt = \langle 2, 1, 4t\rangle$, at the point $(4, 1, 9)$, $u(t = 2) = \langle 2, 1, 8\rangle$.

Meanwhile the velocity of the second curve is $v(s) = dr_2/ds = \langle 2, 0, 2s + 2\rangle$, at $(4, 1, 9)$ we have $v(s = 2) = \langle 2, 0, 6\rangle$.

Thus at $(4, 1, 9)$, the angle between the two curves is given by

$$\theta = \cos^{-1}\left(\frac{u(2)\cdot v(2)}{|u(2)||v(2)|}\right).$$

13.

$$\begin{aligned} L &= \int_0^{\ln 2} \|\mathbf{r}'(t)\| dt \\ &= \int_0^{\ln 2} \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} dt \\ &= \int_0^{\ln 2} \sqrt{(2 + e^{2t} + e^{-2t})} dt = \int_0^{\ln 2} \sqrt{(e^t + e^{-t})^2} dt \\ &= \int_0^{\ln 2} (e^t + e^{-t}) dt = (e^t - e^{-t})|_0^{\ln 2} = 3/2. \end{aligned}$$

14. $r(t) = r(t(s)) := r(s)$. Need to find $t = t(s)$ which is an inverse function of $s = s(t)$. But

$$\begin{aligned} s =& s(t) = \int_0^t \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2 + 64} dt \\ &= \int_0^t \sqrt{4 + 64} dt = \sqrt{68} t \Rightarrow t = t(s) = s/\sqrt{68} \end{aligned}$$

Substituting $t = s/2\sqrt{17}$ into r(t) we obtain the reparametrized equation for the same curve $r = r(s)$, where s denotes the arc length variable:

$$\mathbf{r}(s) = \cos(\frac{s}{\sqrt{17}})\mathbf{i} + \sin(\frac{s}{\sqrt{17}})\mathbf{j} - 4\frac{s}{\sqrt{17}}\mathbf{k}.$$

Now since s is the distance on the curve from $(1, 0, 0)$ to $r(s)$, the point which is $s = 2$ away from (1,0,0) (which corresponds to $s = 0$) is $(\cos(\frac{2}{\sqrt{17}}), \sin(\frac{2}{\sqrt{17}}), -\frac{8}{\sqrt{17}})$.

15* c) Given two lines ℓ_1 and ℓ_2, there exists a vector **AB** perpendicular to both lines ℓ_1 and ℓ_2. If we draw a third line ℓ_3, passing through B and parallel to ℓ_1, then **AB** is also perpendicular to ℓ_3. Therefore **AB** $\perp$ the plane passing through ℓ_2 and ℓ_3.

From the graph we know that AB is the distance between ℓ_1 and the plane passing through ℓ_2, ℓ_3.

But it is easy to find the distance between a line and a plane. In our case, take a point P on the line $\ell_1 : \frac{x-3}{1} = \frac{y-2}{2} = \frac{z+1}{1}$; take another point Q on the line ℓ_2, which is the x-axis. Say, $P = (3, 2, -1), Q = (0, 0, 0), \mathbf{PQ} = (-3, -2, 1)$. The normal direction $\mathbf{N}$ of the plane determined by ℓ_2 and ℓ_3 is given by

$$\mathbf{N} = \mathbf{L_2} \times \mathbf{L_3} = \mathbf{L_2} \times \mathbf{L_1} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -\mathbf{j} + 2\mathbf{k},$$

where $\mathbf{L_2}$ is the direction of ℓ_2, $\mathbf{L_2} = (1,\ 2,\ 1)$ and $\mathbf{L_1}$ is the direction of x-axis, $\mathbf{L_1} = (1,\ 0,\ 0)$.

We have,

$$\text{Distance between } \ell_1 \text{ and the plane}$$

$$= \text{length of } \mathbf{proj_N PQ} = \|\frac{\mathbf{PQ} \cdot \mathbf{N}}{\|\mathbf{N}\|^2}\mathbf{N}\|$$

$$= \frac{|\mathbf{PQ} \cdot \mathbf{N}|}{\|\mathbf{N}\|} = \frac{|(-3, -2, 1) \cdot (0, -1, 2)|}{\sqrt{(-1)^2 + 2^2}} = \frac{4}{\sqrt{5}},$$

which is also the distance between ℓ_1 and ℓ_2.

16.

$$T = \frac{dr}{ds} = \frac{dr/dt}{|dr/dt|}$$

$$\frac{dT}{dt} = \langle -\frac{4t + 18t^3}{(1 + 4t^2 + 9t^4)^{3/2}}, \frac{2 - 18t^4}{(1 + 4t^2 + 9t^4)^{3/2}}, \frac{6t + 12t^3}{(1 + 4t^2 + 9t^4)^{3/2}} \rangle$$

$$N(t) = \frac{1}{\kappa} dT/ds = \frac{dT/dt}{|dT/dt|} \qquad \text{At } (-1, 1, -1) \text{ that is } t = -1,\ N(-1) = \langle 11, -8, -9 \rangle / \sqrt{266}$$

$$\kappa = |\frac{dT}{ds}| = \frac{1}{|v|} |\frac{dT}{dt}|$$

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$$

$$a_T = \frac{d^2 s}{dt^2}$$

$$a_N = \kappa (\frac{ds}{dt})^2 = \frac{\|v\|^2}{R} \quad \text{(R equal to the radius of the osculate circle, or called radius of curvature)}$$

Alternatively, since T, N are unit vectors, we can also first find $\mathbf{a} = \mathbf{r}''(t)$. Then use dot product or projection to get

$$a_T = \mathbf{a} \cdot \mathbf{T}$$
$$a_N = \mathbf{a} \cdot \mathbf{N}.$$