Instructor: Dr. Shijun Zheng

Review Test 2 **Name**
Math 2243 **Id**

Read each question carefully. Avoid simple mistakes. Put a box around the final answer to a question. (Use the back of the page if necessary).
You must show your work to support your answer.

1. Find the differential of $u = \sqrt{x^2 + y^2 + z^2}$.

2. Use the Chain rule (plot a tree diagram) to find the indicated partial derivatives $u = x^2 + \sqrt{y^2 + z^2}$, $x = \sin(r)\cos(s)$, $y = \sin(r)\sin(s)$, $z = 3$. Find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$. Your final answer should be given in terms of r and s only.

3. Find the absolute maximum and absolute minimum values of $f(x, y) = xy - x + y$ on the closed set D which is bounded by the parabola $y = x^2$ and the line $y = 4$.

4. Find the directional derivative of the function at the given point in the direction of $\mathbf{v}$:

$$f(x, y) = x - 2x\sqrt{y}, \quad (2, 9), \quad \mathbf{v} = \langle 1, -1 \rangle.$$

5.
   - Let $f(x, y) = 5xy^2/(x^2 + y^2)$.
     a) Find an equation for the tangent plane to the graph $z = f(x, y)$ at the point $(1, 2, 4)$.
     b) In which direction is f increasing most rapidly at the point $(1, 2)$?
   - Find the tangent plane to the surface $z = y \ln x$ at $(1, 4, 0)$.

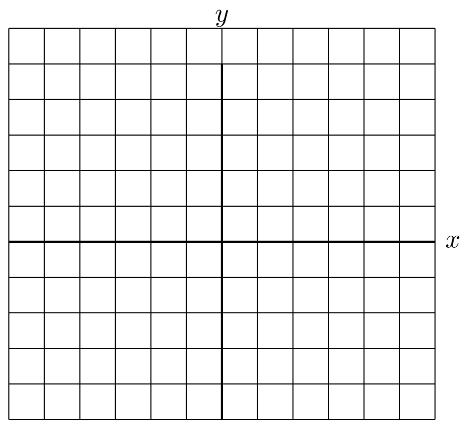
6. a) Evaluate the following limit

$$\lim_{x \to \infty} \left(\frac{x}{\sqrt{x^2 + 1}}\mathbf{i} + \frac{\sin x}{x}\mathbf{j} - \tan^{-1}(x)\mathbf{k} \right)$$

   b) Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \mathbf{i} + \sin t\mathbf{j} - \sqrt{t}\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} - \mathbf{j}$.

7. Let $h(x, y) = \sqrt{9 - x^2 - y^2}$

   (a) Find and sketch the domain of the function h



(b) Find the range of $h(x, y)$

8. Find and sketch the domain of the function

$$f(x, y) = \frac{\sqrt{y - x^2}}{1 - x^2}$$

9. Use the definition of continuity to explain whether or not the function $f(x, y)$ is continuous at $(0, 0)$

$$f(x, y) = \begin{cases} \frac{3xy^2}{x^2+y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

10.
- Use the limit definition to find the partial derivative $\frac{\partial f(x,y)}{\partial y}$ where $f(x, y) = xy^2$.
- Find the indicated partial derivatives: f_{xyy}, where

$$f(x, y) = ye^{\frac{x}{y}}$$

11. * Apply the method of Lagrange multipliers to minimize $f(x, y) = x^2+4y^2$ subject to the constraint $3x + 2y = 60$.

12. Sketch the region of integration, and evaluate the integral:

- [Ex 15.1# 18]

$$\int_0^1 \int_0^{\sqrt{1-s^2}} 8t dt ds$$

- [Ex. 15.1 # 5]

$$\int_0^{\pi} \int_0^{x} x \sin(y) dy dx$$

- $\iint_R x\sin(xy)dA$ where $R = \left\{(x, y)|0 \le x \le 1,\ \ 0 \le y \le \frac{\pi}{4}\right\}$

13. Evaluate the integral $\iint_D y^2 dA$, where D is the region bounded by the upper half of the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $y = x$, and $y = -x$.

14. (optional*) Given a rectangular coordinates $(-1, -2, 3)$, convert into cylindrical coordinates and spherical coordinates respectively.

15. Sketch the region of integration, reverse the order of integration and evaluate the integral:

    - [Ex 15.1# 31]

    $$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y} dydx$$

    - [Ex. 15.1 #32]

    $$\int_0^2 \int_x^2 2y^2 \sin(xy)dydx$$

16. [Ex 15.2 #5] Sketch the region bounded by the given lines and curves. Then find the area using double integral:
    The curve $y = e^x$ and the lines $y = 0, x = 0$, and $x = \ln 2$.

Solutions

1. If $u = u(x, y, z)$, by definition the differential of u is given by $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$. Since

$$\frac{\partial u}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{(x^2 + y^2 + z^2)^{1/2}}$$
$$\frac{\partial u}{\partial y} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2y) = \frac{y}{(x^2 + y^2 + z^2)^{1/2}}$$
$$\frac{\partial u}{\partial z} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z) = \frac{z}{(x^2 + y^2 + z^2)^{1/2}},$$

it follows that

$$du = \frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}}.$$

2. See the diagrams on the section of Chain rules in Chap.14.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

Compute the following partial derivatives and substitute into the above to yield the answer.

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = \frac{y}{\sqrt{y^2+z^2}}, \frac{\partial u}{\partial z} = \frac{z}{\sqrt{y^2+z^2}}$$
$$\frac{\partial x}{\partial r} = \cos(r)\cos(s), \frac{\partial x}{\partial s} = -\sin(r)\sin(s)$$
$$\frac{\partial y}{\partial r} = \cos(r)\sin(s), \frac{\partial y}{\partial s} = \sin(r)\cos(s)$$
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial s} = 0.$$

3. Step 1. Find the critical point(s). Solve $\begin{cases} f_x = y - 1 = 0 \\ f_y = x + 1 = 0 \end{cases}$ to get the solution $(x, y) = (-1, 1)$, where the value of f is $f(-1, 1) = 1$.

Step 2. Find the maximum and minimum of f on the boundary of the set D. Solving the system $\begin{cases} y = x^2 \\ y = 4 \end{cases}$ shows the intersection points are $(-2, 4)$ and $(2, 4)$.

Thus the boundary of D has two pieces: one is the line segment $y = 4, -2 \leq x \leq 2$, the other is part of a parabola $y = x^2, -2 \leq x \leq 2$.

On the line segment the value of f is given by $f(x, 4) = 4x - x + 4 = 3x - 4$, $x \in [-2, 2]$. For the function $3x - 4$, $x \in [-2, 2]$, apparently the maximum is $y_{max} = 3(2) - 4 = 2$, $y_{min} = 3(-2) - 4 = -10$.

On the parabola curve, the value of f (restricted on the parabola) is given by $f(x, x^2) = x(x^2) - x + x^2 = x^3 + x^2 - x$, $x \in [-2, 2]$. For this function $y = x^3 + x^2 - x$, $y' = 3x^2 + 2x - 1 = (x + 1)(3x - 1)$. So the critical points are $x = -1, x = 1/3$. Compare $f(-1) = 1, f(1/3) = -5/27, f(-2) = -2, f(2) = 10$.

Step 3. From Step 2, consider all the possible points above at those critical points or end points, we know that $f_{max} = 10$ and $f_{min} = -2$.

5. (a) The equation of the tangent plane at a given point on $z = f(x, y)$ has the Normal vector

$$N = \langle f_x, f_y, -1\rangle_{(1,2)}$$

Then the equation is given by

$$N \cdot \langle x - 1, y - 2, z - 4\rangle = 0.$$

(b) The direction for which f increases most rapidly at the point $(1, 2)$ is

$$u = \frac{\nabla f}{|\nabla f|}$$

where $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{(1,2)}$.

6. (a) $0\mathbf{i} + 0\mathbf{j} - \frac{\pi}{2}\mathbf{k}$

(b)

$$r(t) = \int r'(t)dt = (\int 1\,dt)i + (\int \sin t dt)j - (\int \sqrt{t} dt)k$$
$$= t\mathbf{i} - \cos t\mathbf{j} - \frac{2}{3}t^{3/2}\mathbf{k} + \mathbf{C}$$

where $\mathbf{C}$ is an arbitrary constant vector. Using the condition $r(1) = i - j$ by sub $t = 1$ into the expression of $r(t)$ above, we can find the value of $\mathbf{C}$.

7. Domain: $\{(x, y) \in R^2 : y \geq x^2, x \neq \pm 1\}$

8. We use two-path test to show f does not have a limit at (0,0), thus f is discontinuous at $(0, 0)$. Take the path $y = kx$, k a constant, then along this path

$$f(x, y) = \frac{3xy^2}{x^2 + y^4} = \frac{3xk^2x^2}{x^2 + k^4x^4} = \frac{3k^2x}{1 + k^4x^2} \to 0 \quad as\ x \to 0$$

But along the other path $y = \sqrt{x}$,

$$f(x, y) = \frac{3xy^2}{x^2 + y^4} = \frac{3x \cdot x}{x^2 + x^2} = \frac{3x^2}{2x^2} \to \frac{3}{2} \quad as\ x \to 0$$

So there are two different limits along different paths, which means the limit of f does not exist at $(0, 0)$.

10. (a) To describe the region R, we know from the outside integration variable s and the inner integration variable t that R is bounded by two vertical lines $s = 0$ and $s = 1$. Then for each fixed s, the limit of t goes from $t = 0$ to $t = \sqrt{1 - s^2}$. So this region R is bounded by two pieces, one is the semi circle $t^2 + s^2 = 1$, the other is $t = 0$, the s-axis.

Now we get the value of the double integral.

$$\int_0^1 \int_0^{\sqrt{1-s^2}} 8tdtds = \int_0^1 \left(4t^2|_{t=0}^{t=\sqrt{1-s^2}}\right) ds$$
$$= \int_0^1 \left(4(1 - s^2)\right) ds = 4 - \frac{s^3}{3}|_{s=0}^{s=1} = 4 - \frac{1}{3} = 11/3.$$

(b) This region R is bounded by $x = 0$, $x = \pi$ and $y = 0$ and $y = x$. So it is a triangle.

$$\int_0^\pi \int_0^x x\sin(y)dydx = \int_0^\pi x\left((-\cos y)|_{y=0}^{y=x}\right) dx$$
$$= \int_0^\pi x(-\cos x + 1)dx = -\int_0^\pi x\cos x dx + \int_0^\pi x dx$$
$$= -(x\sin x + \cos x)|_0^\pi + \frac{x^2}{2}|_0^\pi = 2 + \frac{\pi^2}{2}.$$