Read carefully each problem. Show all your work. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

1 [10] *Solve the system using either Gaussian elimination with back-substiution or Gauss-Jordan elimination.*

a)

$$-x + 2y = 1.5$$
$$2x - 4y = 3$$

b)

$$x_1 + x_2 - 5x_3 = 3$$
$$x_1 \quad\quad - 2x_3 = 1$$
$$2x_1 - x_2 - x_3 = 0$$

2 [10] *Solve the homogeneous linear system corresponding to the coefficient matrix provided:*

a) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3 [10] *a) Write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve the matrix equation for $\mathbf{x}$.*

$$2x_1 + 3x_2 = 5$$
$$x_1 + 4x_2 = 10$$

(b) Solve the matrix equation for a, b, c, d

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix}$$

4 [10] *a) If $AB = 0$, is it necessarily $A = 0$ or $B = 0$? Consider the example $\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{pmatrix}$*

b)Show that if $AB = 0$ and A is invertible, then $B = 0$.

5 [10] *Find the inverse of the matrix (if it exists).*

a) $\begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & -9 \\ 7 & 16 & -21 \end{pmatrix}$　　　　d) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{pmatrix}$

6 [10] *Prove that if $A^2 = A$, then $I - 2A = (I - 2A)^{-1}$.*

7 [10] *Let A be an n by n matrix. Which of the following statements are equivalent to the statement that A is invertible?*

(1) *A is singular*

(2) *There exists a matrix B such that $BA = I_n$*

(3) *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every n by 1 column matrix $\mathbf{b}$*

(4) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution*

(5) *A is row-equivalent to I_n*

(6) *A is column-equivalent to I_n*

(7) *A can written as the product of elementary matrices.*

(8) *Determinant of A is nonzero*

(9) *$A\mathbf{x} = \mathbf{0}$ has infinitely many solutions*

8 [10] *Factor the matrix into a product of elementary matrices.*

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$C = \begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$

9 [10] *(optional) Solve the system $Ax = b$ by*

    1) *finding the LU-factorization of the coefficient matrix A*
    2) *solving the lower triangular system $Ly = b$, and*
    3) *solving the upper triangular system $Ux = y$.*

a)

$$2x + y = 1$$
$$y - z = 2$$
$$-2x + y + z = -2$$

b)

$$2x_1 = 4$$
$$-2x_1 + x_2 - x_3 = -4$$
$$6x_1 + 2x_2 + x_3 = 15$$
$$-x_4 = -1$$

c)

$$x_1 - 3x_2 = -5$$
$$x_2 + 3x_3 = -1$$
$$2x_1 - 10x_2 + 2x_3 = -20$$

10 [10] *Let A be a nonsingular matrix. Prove that a) if B is row-equivalent to A, then B is also nonsingular. b) Use $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$ to show that A^T is also invertible.*

11[10] *(optional) Using a system of equations to write the partial fraction decomposition of the rational expression. Then solve the system using matrices.*

$$\frac{4x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

where A, B, C are constants.

12 [10] *Give three distinct examples of elementary matrices and explain how they correspond to row operations for a given matrix of 3 by 3.*

Solutions

1 (a) No solutions.

1 (b) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, *where t is a parameter running through* $(-\infty, \infty)$.

2 (a) The homogeneous system $Ax = 0$ means

$$x_1 + x_4 = 0$$
$$x_3 = 0$$
$$0 = 0$$

Hence the solution is $x_1 = -t, x_2 = s, x_3 = 0, x_4 = t$, where $-\infty < s, t < \infty$.

2 (b) The system is equivalent to $0x_1 + 0x_2 + 0x_3 = 0$ Therefore, we have the freedom of choosing values of the unknown variables. $x_1 = s, x_2 = t, x_3 = r$, where the parameters s, t, r can be any real numbers.

3. (a) $(x_1, x_2) = (-2, 3)$.

(b) Multiplying the equation both sides by $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$ *on the right, we have*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1}$$

Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 48 & -31 \\ -7 & 6 \end{pmatrix}$$

4 a) No. b) We need to show if A^{-1} exists and $AB = 0$, then $B = 0$. Multiplying A^{-1} on the left on both sides of the equation $AB = 0$, we have

$$A^{-1}AB = A^{-1} \cdot 0,$$

that is, $B = 0$ (since $A^{-1}A = I$ and $I \cdot B = 0$)

5. b) Row operation $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix} \xrightarrow{(-3)R1 + R2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix}$ *which shows that one of the rows has all zeors, thus it is Not row equivalent to identity matrix. Hence the matrix is Not invertible.*

c) Perform row operation on the adjoining matrix $\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -9 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{pmatrix}$ $\xrightarrow{(-3)R1+R2}$

$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{pmatrix}$ $\xrightarrow{(-7)R1+R3}$ $\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 0 & 2 & -14 & -7 & 0 & 1 \end{pmatrix}$ $\xrightarrow{(-2)R2+R3}$

$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{pmatrix}$ $\xrightarrow{(-3)R3+R2}$ $\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{pmatrix}$ $\xrightarrow{(-1/2)*R3}$

$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$ $\xrightarrow{R3+R1}$ $\begin{pmatrix} 1 & 2 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$ $\xrightarrow{(-2)*R2+R1}$ $\begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & -13 & \frac{11}{2} \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$

Hence $A^{-1} = \begin{pmatrix} \frac{3}{2} & -13 & \frac{11}{2} \\ 0 & 7 & -3 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$

d) Not invertible.

6. Proof. Since $A^2 = A$, we have

$$(I - 2A)^2 = I - 4A + 4A^2$$
$$= I - 4A + 4A = I,$$

which shows that $(1-2A)^{-1} = I-2A$ by definition of an inverse matrix.
□

7. The fact that A is invertible is equivalent to $(2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8)$.

Basically this question asks the student to describe an invertible matrix A (using alternative statements). This is a logic-related type of question to help the understanding and getting more familiar with certain concepts in LA.

For instance, (2) is an equivalent statement to suggest A is invertible, because: if $BA = I$, then $AB = I$ by the fact A factors into product of elementary matrices. Then by definition, this means A is invertible, whose inverse is given by B.

(3) $Ax = b$ has a unique solution for each vector b. This statement also means that A is invertible, because if A is not invertible, then one can row reduces A to a matrix having an echelon form that is NOT the identity matrix. More specifically the echelon form will have at least on the bottom line all zeros.

At this point you can take b vector to have on the last entry value equal to 1, say. Then you will find immediately the system $Ax = b$ with such b will have no solution.

The above argument suggests that A must be invertible (or equivalently, A can be row-reduced to identity matrix).

8. Do a finite sequence of row operations to convert A to I. This corresponds to $E_k \cdots E_1 A = I$ for certain elementary matrices E_i. Record the E_i corresponding to each row operation. Then $A = E_1^{-1} \cdots E_k^{-1}$ is the factorization.

8. (a)

$$A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

(c). Row operation $C = \begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{pmatrix} \overset{R1 \leftrightarrow R4}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 4 & 0 & 0 & 2 \end{pmatrix} \overset{(-4)R1+R4}{\longrightarrow}$

$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 10 \end{pmatrix} \overset{\frac{1}{10}R4}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{(-2)R4+R3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{(-1)R4+R2}{\longrightarrow}$

$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{2R4+R1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{(-1)*R3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

From the above sequence we keep track the corresponding elementary

matrices at each step $E_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix}$; $E_3 = $

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{pmatrix}$ $E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $E_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $E_6 = $

$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $E_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Then $A = E_1^{-1} \cdots E_7^{-1}$ *where* $E_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$; $E_3^{-1} = $

$$E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad E_6^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_7^{-1} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

9. $A = LU$. First solve $Ly = b$ then solve $Ux = y$, this process can be expressed via matrix notion

$$x = U^{-1}(L^{-1}b)$$

if both U and L are invertible. If not, then we just solve the two equations directly (and separately) by row-echelon.

10. a) Proof. B is row equivalent to A means that one can do a finite sequence of elementary row operation to convert A to B (or B to A). Since each row operation amounts to multiplication on the left by an elementary matrix E_i, we know that B can be written as $B = E_k \cdots E_1 A$. Now B is invertible (or nonsingular) because of the existence of $B^{-1} = A^{-1} E_1^{-1} \cdots E_k^{-1}$.

b) The problem asks to prove that if A is invertible, then A^T is also invertible. Hence we need to show the existence of the inverse of A^T.

Claim. The inverse of A^T is $(A^{-1})^T$. In fact,

$$(A^{-1})^T (A^T) = (AA^{-1})^T = I^T = I$$

also

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Therefore the claim is proved true.

11. First find the common denominator $(x-1)(x+1)^2$. The equation then becomes

$$\frac{4x^2}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

Compare the coefficients of the x^2, x and constant terms for the numerator, we obtain three linear equations with three unknowns A, B, C. Solve the linear equations for A, B, C.

12. For example $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Multiplied by E_1, E_2, E_3 on the left of a given matrix A correspond to row operations: exchanging the first and second rows, multiplying the third by 3 and Row 2 add $(-2)\times$ Row 1$\rightarrow$ Row 2, respectively for A.