Math 3230 Review Test I

It is necessary to show your work in order to receive credits or partial credits.

1. Verify that the equation $y^2 = x^2 - C$, where C is a constant, satisfies the differential equation
$$\frac{dy}{dx} = \frac{x}{y}$$

2. Find the exact solution of the initial value problem
$$\frac{dy}{dx} = y^2 \cos x, \quad y(0) = 1$$

3. Find the solution to the initial value problem
$$x' = x \sin t + 2te^{-cost}, \quad x(0) = 1$$

4. Determine if the equation is exact and solve it if it is.
$$(2x \sin y + 3x^2 y)dx + (x^3 + x^2 \cos y + y^2)dy = 0$$

5. (a) Determine if the differential equation
$$ydx + (y - x)dy = 0$$
is homogeneous. If so, determine its degree. (b) Solve the equation.

6. * Find a general solution of
$$\frac{dy}{dx} = 6\frac{y}{x} - xy^2$$

7. * Use the Existence and Uniqueness Theorem for linear IVPs to determine the largest interval on which the solution is guaranteed to exist.
$$y' + \frac{2y}{x^2 - 9} = \frac{x}{x^2 - 9}, \; y(4) = -3$$

8. Figure 1 (Page 3) is the direction field for the differential equation $y' = y(y-1)(y+1)$.
   (a) Draw on the direction field the solutions of the differential equation satisfying each of the following initial values.
      i. $y(0) = 0.0$
      ii. $y(0) = 0.5$

iii. $y(0) = -1.5$

(b) For the solution $y(t)$ with initial condition $y(0) = 0.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

(c) For the solution $y(t)$ with initial condition $y(0) = -1.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

9. Figure 2 (Page 3) is the direction field for the differential equation $y' = y - t$.

(a) Draw on the direction field the solutions of the differential equation satisfying each of the following initial values.

i. $y(0) = 0.0$

ii. $y(0) = 1.0$

iii. $y(0) = -1.0$

iv. $y(0) = 2.0$

(b) Are there any constant solutions $y = c$ to this differential equation? If so, show them on the direction field.

(c) Are there any straight line solutions $y = mt + b$? If so indicate them on the direction field.

(d) There is a number c such that all solutions with initial condition $y(0) > c$ satisfy $\lim_{t\to\infty} = \infty$ and all solutions with initial condition $y(0) < c$ satisfy $\lim_{t\to\infty} = -\infty$. Find this number c by inspecting the direction field.

Figure 1: Direction Field for Exercise 8

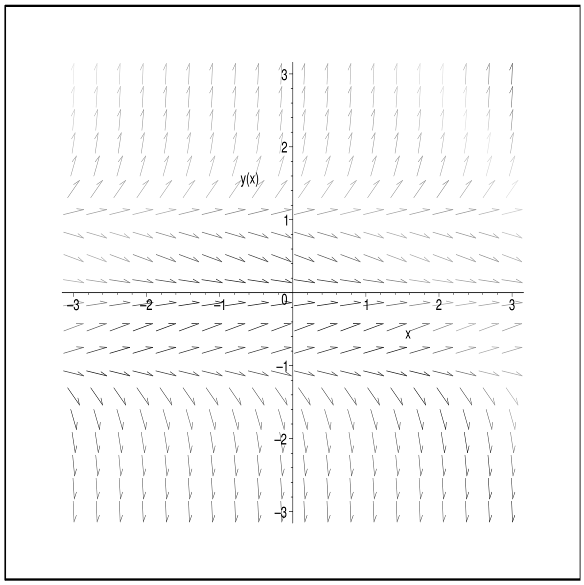
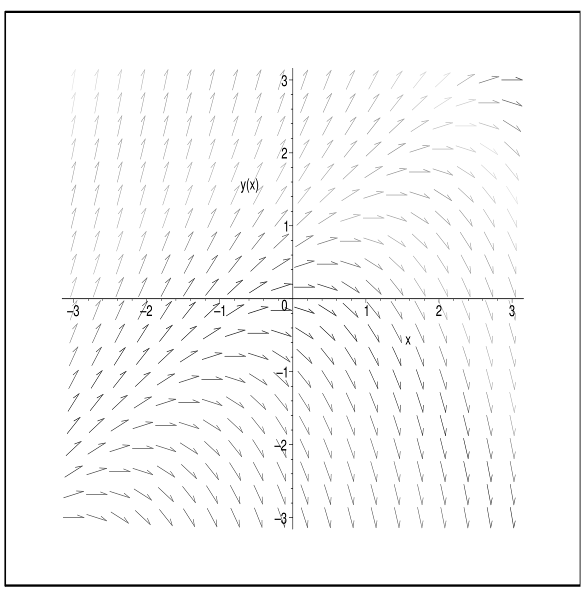


Figure 2: Direction Field for Exercise 9



10. Determine if each of the following equations is separable (yes or no), and/or linear (yes or no). Do **not** solve the equations!!

    (a) $y' = y^2 - t$
    (b) $t^2 y' = 1 - 2ty$
    (c) $yy' = 3 - 2y$
    (d) $\dfrac{y'}{y} = y - t$
    (e) $ty' = y - 2ty$
    (f) $(t^2 + 3y^2)y' = -2ty$
    (g) $y' = ty^2 - y^2 + t - 1$
    (h) $t + y' = y - 2ty$

11. Solve each of the following initial value problems. You **must** show your work.

    (a) $y' = 2y + 5e^{2t}, \quad y(0) = -1.$
    (b) $y' = y^2 t^3, \quad y(1) = -1.$
    (c) $y' + 3y = 4e^{-3t}\sin 2t, \quad y(0) = -1.$
    (d) $y' + \dfrac{3}{t}y = 7t^3, \quad y(1) = -1.$

12. * Newton's law of cooling states that the rate at which a body cools (or heats up) is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. A turkey which is initially at room temperature (70° F) is placed in a 350° F oven at time $t = 0$. Write an initial value problem which is satisfied by the temperature $T(t)$ of the turkey at time t.

13. * A tank contains 100 gal of brine made by dissolving 80 lb of salt in water. Pure water runs into the tank at the rate of 4 gal/min, and the mixture, which is kept uniform by stirring, runs out at the same rate. Find the amount $y(t)$ of salt in the tank at any time t.

Solutions

1. Regard $y^2 = x^2 - C$ as an equation that implicitly defines $y = y(x)$ as a function of x. Taking derivative in x both sides by Chain rule yields

$$2y\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}.$$

2. Separate variables to obtain

$$\begin{aligned}
\frac{dy}{y^2} &= \cos x dx \\
\int \frac{dy}{y^2} &= \int \cos x dx \\
-y^{-1} &= \sin x + C \\
y &= \frac{-1}{\sin x + C}
\end{aligned}$$

In order to determine C, you will need to plug in the initial condition $y(0) = 1$.

3. This is first order linear ODE $x' + Px = Q$, where $P = -\sin t, Q = 2te^{-cost}$. The general formula gives

$$\begin{aligned}
x(t) &= e^{-\int P} \int e^{\int P} Q dt = e^{\int (\sin t)} \int e^{\int (-\sin t)} 2te^{-cost} dt \\
&= e^{\int (\sin t)} \int e^{\cos t} 2te^{-cost} dt = e^{\int (\sin t)} \int 2t dt \\
&= e^{-cost}(t^2 + C)
\end{aligned}$$

Now plugging in $t = 0, x = 1$ to obtain $C = e$.

4. It is Exact by the following test: The differential form $Mdx + Ndy = 0$ is exact $\iff$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Since it is exact, there exists $f(x, y)$ such that $df = f_x dx + f_y dy = Mdx + Ndy$. We will solve f to obtain the equation $f(x, y) = C$ which implicitly defines the solution of

$$(2x \sin y + 3x^2 y)dx + (x^3 + x^2 \cos y + y^2)dy = 0$$

From $\frac{\partial f}{\partial x} = 2x \sin y + 3x^2 y$ we get

$$f(x, y) = \int (2x \sin y + 3x^2 y)dx = x^2 \sin y + x^3 y + C(y)$$

Taking derivative in y of the above yields

$$\begin{aligned}
&\partial_y f(x, y) = \partial_y (x^2 \sin y + x^3 y + C(y)) \\
&= x^2 \cos y + x^3 + C'(y) = N = x^3 + x^2 \cos y + y^2
\end{aligned}$$

which suggests $C'(y) = y^2 \to C(y) = y^3/3$. Hence we arrive at the equation

$$f(x, y) = x^2 \sin y + x^3 y + y^3/3 = C.$$

5. a) The standard equation is $Mdx + Ndy = 0$. Compare and we see that $M(x,y) = y$ and $N(x,y) = y - x$. This is homogeneous of degree $k = 1$.

   (b) Sub $x = yv \longrightarrow dx = ydv + vdy$.

$$
\begin{aligned}
&y(ydv + vdy) + (y - yv)dy = 0 \\
&y^2dv + ydy = 0 \\
&ydv + dy = 0 \\
&\text{Substitute back } v = x/y \text{ to obtain} \\
&x = y\ln|y| + Cy \\
\textit{or } &y = C_2 e^{x/y}
\end{aligned}
$$

   where C_2 is an arbitrary number.

6. This is first-order quadratic equation (Bernoulli type). $n = 2$ Substitution $w = y^{1-n} = y^{-1} \to y = w^{-1}$. We have $\frac{dy}{dx} = -w^{-2}\frac{dw}{dx}$ and so

$$
\begin{aligned}
&-w^{-2}\frac{dw}{dx} = 6\frac{w^{-1}}{x} - xw^{-2} \\
&(\text{multiplying } -w^2 \text{ both sides} \to) \quad \frac{dw}{dx} = -6\frac{w}{x} + x
\end{aligned}
$$

   This is a 1st-order ODE, you can solve to get $w = w(x)$ and then replace w by y^{-1} and then simplify to obtain the solution $y = y(x)$.

   Indeed,

$$
\begin{aligned}
w = w(x) &= e^{-\int \frac{6}{x}} \left(\int e^{\int \frac{6}{x}} x dx \right) \\
&= e^{-6\ln|x|} \left(\int x^6 x dx \right) = x^{-6}(x^8/8 + C) \\
&= x^2/8 + Cx^{-6}.
\end{aligned}
$$

   From this we obtain $y = \frac{1}{x^2/8 + Cx^{-6}}$.

7. By the Existence and Uniqueness Theorem for first order linear ODE in Section 2.1, we know that the interval of existence is where both P and Q are continuous. Here $P = \frac{2}{x^2-9}, Q = \frac{x}{x^2-9}$ for which there are three open intervals $(-\infty, -3), (-3, 3), (3, \infty)$ on which P and Q are continuous simultaneously. However , only on $(3, \infty)$ can defines a solution $y = y(x)$ whose graph passes $(4, -3)$. So the largest interval must be $(3, \infty)$.

10. (a) Not separable, nonlinear
    b) not separable, linear

c) separable, nonlinear
d) not separable, nonlinear
e) separable, linear
f) not separable, nonlinear
g) separable, nonlinear. In fact,

$$y' = (t-1)y^2 + (t-1)$$
$$y' = (t-1)(y^2+1)$$

You see that the r.h.s is a product of $f(t)g(y)$.
h) not separable, linear

11. (a)

$$y(t) = e^{\int 2dt} \int e^{\int(-2)dt} 5e^{2t} dt$$
$$= e^{-2t}(5t + C)$$

Plug in $y(0) = -1$ to get $C = -1$.

(b) Separate variables and then integrate both sides to get

$$\int y^{-2} dy = \int t^3 dt$$
$$- y^{-1} = t^4/4 + C$$
$$\therefore\ y = (-t^4/4 - C)^{-1}$$

Plugging in $t = 1, y = -1 \to y = (-t^4/4 - 3/4)^{-1}$

(c)

$$y(t) = e^{-\int p}\left(\int e^{\int p} Q dt + C\right)$$
$$= e^{-\int 3dt} \int e^{\int 3dt} 4e^{-3t} \sin 2t dt$$
$$= e^{3t} \int 4 \sin 2t dt = 2e^{3t}(-\cos 2t + C)$$

The I.C. $y(0) = -1$ then gives $C = 1/2$.

(d)

$$y(t) = e^{-\int \frac{3}{t} dt} \int e^{\int 3/t dt} 7t^3 dt$$
$$= t^{-3} \int t^3 7t^3 dt = t^{-3}(t^7 + C) = t^4 + Ct^{-3}$$

The I.C. $(1, -1)$ gives $C = -2$.

12. Newton's law of heating: let $T = T(t)$ be the temperature at time t.

$$\frac{dT}{dt} = (350 - T)$$
$$T(0) = 70$$

13. Assume that the volume of brine is constantly 100 gal any time. Denote $y = y(t)$ the amount of salt at time t. When initially $t = 0$, $y(0) = 80$ (lb). Then every minus $(\Delta t = 1)$ the rate of change is $\Delta y = -4\times$ density of brine at time $t = -4\frac{y(t)}{100}$. So the model is given by the equation

$$\frac{\Delta y}{\Delta t} = -4\frac{y}{100}$$
$$\frac{dy}{dt} = -4\frac{y}{100}$$