Math 3230
Review Exercises for Exam 2

It is necessary to show your work in order to receive credits or partial credits.

1. a) Find a fundamental set of solutions of $y'' - 4y' + 9y = 0$ and write the general solution.
   b) Explain why the two solutions you find in part (a) are linearly independent.

2. a) Find a fundamental set of solution of $y''' + y = 0$. b) Explain why the three solutions you find in part (a) are linearly independent (using the Wronskian).

3. [optional*] Let m, k, a, b be constants. Solve the (second-order linear) IVP

$$\begin{cases} mx''(t) + kx(t) = 0 \\ x(0) = a, x'(0) = b \end{cases}$$

4. Use the method of undetermined coefficients to find the solution to IVP

$$y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, y'(0) = 2$$

5. Use the method of undetermined coefficients to find the general solution to the equation

$$y'' + y' - 2y = 8\sin 2t$$

6. Find the solution to the initial value problem

$$x' = x\sin t + 2te^{-cost}, \quad x(\pi) = -1$$

7. Solve

$$x^2\frac{d^2u}{dx^2} + 7x\frac{du}{dx} - 7u = 0$$

8. Use variation of parameters to find the general solution to the equation

$$x'' + x = \csc t$$

9. * Find a general solution of

$$\frac{dy}{dx} = -\frac{y}{x} - xy^2$$

10. * Use the Existence and Uniqueness Theorem for linear IVPs to determine the largest interval on which the solution is guaranteed to exist.

$$y' + \frac{2y}{x^2 - 9} = \frac{x}{x^2 - 9}, \; y(4) = -3$$

11. a) State the Definitions of Linear Dependent and Linear Independent for $S = \{f_1, f_2, \dots, f_n\}$, a set of n functions.
b) State the Principle of Superposition
c) (Existence and Uniqueness for second-order) Suppose $p(t), q(t)$ and $f(t)$ are ________ on an open interval (a, b) containing $t = t_0$. Then the IVP

$$y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = y_0, y'(t_0) = y_1$$

has a ________ solution on (a, b). (Hint: Read Chap.4)

12. Use the method of variation of parameters to find a general solution to the equation $y'' - 2y' + y = e^t \ln t$.

13. Find a general solution to the equation $y''' - y'' + 4y' - 4y = 0$

14. Write the following equation for $y = y(x)$ as a system of first-order, linear differential equations

$$y''' + 3y'' + 6y' + 3y = x$$

15. Write the following initial value problem as a system of first-order, linear differential equations in matrix notation

$$y'' + 2y' + 4y = 3\cos 2t, \qquad y(0) = 1, y'(0) = 0$$

16. * Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

17. Determine whether $\Phi(t) = \begin{pmatrix} \cos t + 2\sin t & 2\cos t - \sin t \\ \sin t & \cos t \end{pmatrix}$ is a fundamental matrix for the linear system $X'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X(t)$

18. a) Verify $\Phi(t) = \begin{pmatrix} 3e^{4t} & e^{3t} \\ 2e^{4t} & e^{3t} \end{pmatrix}$ is a fundamental matrix for the system $X'(t) = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} X(t)$
b) Find a general solution to the system in (a)
c) Find the solution that satisfies initial condition $X(0) = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

19. Find the general solution to the system

$$Y' = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} Y$$

20. * Use the method of undetermined coefficients to find the solution of the system

$$Y' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} Y + \begin{pmatrix} 0 \\ -2t \end{pmatrix}$$

21. The eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are $\lambda = 1, 2, 3$ and $(0,0,1)^T$, $(1,2,1)^T$, and $(1,3,0)^T$, respectively. Consider the IVP

$$X' = AX, \quad X(0) = X_0$$

where $X_0 = (3,7,4)^T$. a) Find a fundamental set of solutions to the 3 by 3 linear system $X' = AX$, and verify the linear independency using the Wronskian. b) Solve the IVP.

22. Use the method of undetermined coefficients to find a general solution for

$$Y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y + \begin{pmatrix} t \\ -1 \end{pmatrix}$$

23. Use variation of parameters to find a general solution for

$$X' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

24. * Consider the system

$$Y' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} Y \qquad -\infty < t < \infty$$

(a) Find a real-valued fundamental set of solutions of general solution. (b) Identify its equilibrium solution, classify the type and stability characteristics of the equilibrium point.

25. A 3-lb weight stretches a spring 3 in. The mass is raised 1 in above its equilibrium position and then set in motion with a downward velocity of 2 ft/sec, and if there is no damping, determine the displacement function of the mass at any time t.

26. An object of mass 1 lb-s^2/ft is attached to a spring with spring constant $k = 1.25$ lb/ft and is subject to a resistive force $F_r = 2dx/dt$. Determine the displacement of the mass if the object is released from the equilibrium position with an initial velocity of 3 ft/sec in the downward direction. (Hint: draw a picture with coordinates)

27. Answer the following questions concerning the differential equation

$$(*) \qquad t^2 y'' + t y' - y = 6t^2.$$

(a) Verify that $\varphi_1(t) = t$ and $\varphi_2(t) = \dfrac{1}{t}$ are solutions of the associated homogeneous differential equation $t^2y'' + ty' - y = 0$.

(b) Verify that $y_p(t) = 2t^2$ is a solution to equation $(*)$.

(c) What is the general solution of equation $(*)$?

(d) What is the solution of the initial value problem

$$(**) \qquad t^2y'' + ty' - y = 6t^2, \quad y(1) = -1, \quad y'(1) = 1?$$

(e) What is the largest interval on which the initial value problem $(**)$ is guaranteed to have a solution by the existence and uniqueness theorem? Is this answer consistent with the solution that you found in the previous part of this exercise?

Solution Key

1 (a) The characteristic equation is $r^2-4r+9 = 0$ from which we get $r_1 = 2\pm\sqrt{5}i$. Hence the fundamental set is $\{y_1, y_2\} = \{e^{2t}\cos\sqrt{5}t, e^{2t}\sin\sqrt{5}t\}$ The general solution is $y = e^{2t}(C_1\cos\sqrt{5}t + C_2\sin\sqrt{5}t)$

(b) The two solutions $y_1(t), y_2(t)$ are linear independent because they are not constant multiple of one another. Alternatively the Wronskian

$$\begin{vmatrix} e^{2t}\cos\sqrt{5}t & e^{2t}\sin\sqrt{5}t \\ e^{2t}(2\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t) & e^{2t}(2\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t) \end{vmatrix} = 2\sqrt{5}e^{4t} \neq 0$$

$\Rightarrow$linear independent

2 (a) The characteristic equation for $y''' + y = 0$ is

$$r^3 + 1 = 0 \qquad (r+1)(r^2 - r + 1) = 0$$
$$\longrightarrow r_1 = -1, r_2 = \frac{1+\sqrt{3}i}{2}, r_3 = \frac{1-\sqrt{3}i}{2}$$

$\therefore$ Fundamental set $\{y_1(t) = e^{-t}, y_2(t) = e^{t/2}\cos(\frac{\sqrt{3}}{2}t), e^{t/2}\sin(\frac{\sqrt{3}}{2}t)\}$.

(b) We show that these three functions are linear independent.

Assume

$$C_1e^{-t} + C_2e^{t/2}\cos(\frac{\sqrt{3}}{2}t) + C_3e^{t/2}\sin(\frac{\sqrt{3}}{2}t) = 0. \tag{1}$$

Want to show: $C_1 = C_2 = C_3 = 0$. Multiplying $e^{-t/2}$ both sides of (1) we obtain

$$C_1e^{-3t/2} + C_2\cos(\frac{\sqrt{3}}{2}t) + C_3\sin(\frac{\sqrt{3}}{2}t) = 0. \tag{2}$$

Claim. $C_1 = 0$. Otherwise let $t \to -\infty$ and we have the first term going to infinity. Meanwhile $C_2 \cos(\frac{\sqrt{3}}{2}t) + C_3 \sin(\frac{\sqrt{3}}{2}t$ is bounded by $|C_2| + |C_3|$ such that (2) cannot vanish for $t \to -\infty$. So C_1 must be zero.
Now

$$C_2 e^{t/2} \cos(\frac{\sqrt{3}}{2}t) + C_3 e^{t/2} \sin(\frac{\sqrt{3}}{2}t) = 0$$
$$\Longleftrightarrow C_2 \cos(\frac{\sqrt{3}}{2}t) + C_3 \sin(\frac{\sqrt{3}}{2}t) = 0.$$

Substituting $t = 0$ above yields C_2, from which follows

$$C_3 \sin(\frac{\sqrt{3}}{2}t) = 0 \quad \text{for all } t$$

This immediately suggests that $C_3 = 0$. Q.E.D
Remark. Alternatively for (b), to show the linear independency one can also use Wronskian.

4 Step 1. Find the fundamental solution of $y'' + 4y = 0$.

$$r^2 + 4 = 0 \longrightarrow r_{1,2} = \pm 2i$$
$$y_1 = \cos 2t, \; y_2 = \sin 2t \longrightarrow y_{homog} = C_1 \cos 2t + C_2 \sin 2t$$

Setp 2. Find a particular solution to $y'' + 4y = t^2 + 3e^t \quad (*)$.
The term $f(t)$ on the right hand side of the ODE suggests that

$$y_p = (c_0 + c_1 t + c_2 t^2) + Ce^t$$

Plug in this expression of y_p into the ODE $(*)$ we can get four linear algebraic equations which are easy to solve and so we determine these constants.
Step 3. Finally we can apply the I.C. $\quad y(0) = 0, y'(0) = 2$ to $y = y_h + y_p$ to determine C_1, C_2.
[Solution] for Step 1 and 2.

$$y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$$
$$f(t) = t^2 + 3e^t \longrightarrow t^2, t, 1, e^t$$
$$y_p(t) = A + Bt + Ct^2 + De^t$$
$$y_p(t) = -1/8 + (1/4)t^2 + (3/5)e^t$$
$$y(t) = C_1 \cos(2t) + C_2 \sin(2t) - 1/8 + (1/4)t^2 + (3/5)e^t$$

5 We use undetermined coefficients to find the general solution to the equation

$$y'' + y' - 2y = 8 \sin 2t. \tag{3}$$

Step 1. Solve $y'' + y' - 2y = 0$ (homogeneous equation first).
The characteristic equation reads

$$\begin{aligned} & r^2 + r - 2 = 0 \\ & (r-1)(r+2) = 0 \longrightarrow r_1 = 1, r_2 = -2 \\ \therefore & \text{fundamental set } S = \{e^t, e^{-2t}\} \\ & y_h(t) = C_1 e^t + C_2 e^{-2t} \end{aligned}$$

Step 2. Find a particular solution y_p. Since the right hand side has the form $\sin 2t$, the solution $y_p = A\cos 2t + B\sin 2t$. Substituting this expression into (3) yields a system of two linear equation for A and B.
Step 3. Finally the general solutions are given by $y(t) = y_h(t) + y_p(t)$.

6 Write the equation as $x' - x\sin t = 2te^{-cost}$ Then $p(t) = -\sin t$, and so $\mu = e^{\int p(t)} = e^{\cos t}$

7 Solve the Cauchy-Euler equation

$$x^2\frac{d^2u}{dx^2} + 7x\frac{du}{dx} - 7u = 0$$

Step 1. Change of variable $x = e^t \to u(x) = u(e^t) := u(t)$, with $t = \ln x$, $x > 0$. With this substitution the general C-E equation $ax^2u''(x) + bxu'(x) + cu(x) = 0$ is converted to

$$au''(t) + (b-a)u'(t) + cu(t) = 0$$

For our problem $a = 1, b = 7, c = -7$ thus we have

$$u''(t) + 6u'(t) - 7u(t) = 0$$

The characteristic equation is $r^2 - 6r - 7 = 0$, or, $(r+1)(r-7) = 0 \longrightarrow r_1 = -1, r_2 = 7$. This yields the fundamental set $\{e^{-t}, e^{7t}\}$. Backsubstituting $t = \ln x$ gives us

$$S = \{x^{-1}, x^7\}.$$

Hence the general solutions are given by $u(x) = \frac{C_1}{x} + C_2x^7$.

8 Let $x(t) = u_1f_1 + u_2f_2$, where f_1, f_2 are fundamental solutions of $x'' + x = 0$. Then $u_1'(t) = -g(t)f_2/W$, $u_2'(t) = g(t)f_1/W$, where $g = \csc t = 1/\sin t$ and $W = W(f_1, f_2)$ is the Wronskian of f_1, f_2.

9 Let $w = y^{1-n}$, $n = 2$. Then $y = w^{-1}$, $dy/dx = -\frac{1}{w^2}\frac{dw}{dx}$. Substituting these into the Bernoulli equation yields

$$\frac{dw}{dx} - \frac{1}{x}w = x \tag{4}$$

This is a first-order linear ODE for $w = w(x)$. The integrating factor $\mu = e^{\int P} = e^{\int (-1/x)}dx = 1/x$. Multiplying $1/x$ on both sides of (4) yields

$$
\begin{aligned}
&(\frac{1}{x}w)' = 1 \\
&\frac{1}{x}w = x + C \\
&w = x^2 + Cx \longrightarrow y(x) = \frac{1}{x^2 + Cx}.
\end{aligned}
$$

10 The Existence and Uniqueness Theorem for first-order linear ODE

$$y' + b(x)y = f(x),$$

with initial value condition $y(x_0) = y_0 \quad (I.C.)$ says that if $b(x)$ and $f(x)$ are both continuous on the same open interval (a, b) containing x_0, then there exists a unique solution $y = y(x)$ to this ODE defined in (a, b) that satisfies the I.C. Apply this E and U theorem to

$$y' + \frac{2y}{x^2 - 9} = \frac{x}{x^2 - 9}, \ y(4) = -3.$$

We see that $b(x) = \frac{2}{x^2-9}$ and $f(x) = \frac{x}{x^2-9}$ are both continuous on $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. But the only interval that contains $x_0 = 4$ is $(3, \infty)$. So the largest interval on which the solution is guarantee to exist is $(3, \infty)$. Q.E.D.

17 Let $\Phi(t) = \begin{pmatrix} X_1(t) & X_2(t) \end{pmatrix}$, where $X_1(t) = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$ and $X_2(t) = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$. Substituting X_1 to the equation (system) on the left and on the right of

$$X'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X(t)$$

we find that the left equals the right for all t. This verifies $X_1(t)$ is a solution. Similar result holds for $X_2(t)$.
In order to show Φ is a fundamental matrix, we also must show that the two columns $X_1(t), X_2(t)$ are linear independent. Indeed the Wronskian of Φ

$$\det(\Phi(t)) = 3e^{4t}e^{3t} - e^{3t}(2e^{4t}) = e^{7t} > 0$$

which proves the linear independency and so Φ is a fundamental matrix for the given system.
18 a) This works the same as in 17 by writing $\Phi(t) = \begin{pmatrix} X_1(t) & X_2(t) \end{pmatrix}$, where $X_1(t) = \begin{pmatrix} 3e^{4t} \\ 2e^{4t} \end{pmatrix}$ and $X_2(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$.

b) A general solution is given by

$$X(t) = C_1 \begin{pmatrix} 3e^{4t} \\ 2e^{4t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

c) Plug in the initial condition to determine the constants C_i. We have with $t = 0$

$$\begin{pmatrix} 6 \\ -4 \end{pmatrix} = C_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and therefore solve this linear system to get $C_1 = 10, C_2 = -24$.
19 This system is in the case of repeated roots, and there is an example in Chap.6 showing how to find the two linear independent solutions.
Step 1. Solve $|A - \lambda| = 0$ to get $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0 \longrightarrow \lambda_1 = \lambda_2 = -2$
Step 2. Solve $(A - \lambda)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Step 3. Find two linear independent solutions $\Phi = (\Phi_1, \Phi_2)$. $\Phi_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\Phi_2(t) = te^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{w}$ where $\mathbf{w}$ solves $(A - \lambda)\mathbf{w} = \mathbf{v}$
Step 4. Finally the solution of the given system is

$$Y(t) = \Phi(t)\mathbf{C} = C_1\Phi_1(t) + C_2\Phi_2(t)$$

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$$|A - \lambda| = \det \begin{pmatrix} -\lambda & 1 & 0 \\ -6 & 5 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda - 1)(\lambda - 2)(\lambda - 3)$$

$\therefore$

$$X(t) = \Phi(t)C = \begin{pmatrix} e^t\mathbf{v}_1 & e^{2t}\mathbf{v}_2 & e^{3t}\mathbf{v}_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

here $\mathbf{v}_1 = (0, 0, 1)^T$, $\mathbf{v}_2 = (1, 2, 1)^T$, and $\mathbf{v}_3 = (1, 3, 0)^T$.
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$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$|A - \lambda| = \det \begin{pmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{pmatrix} = \lambda^2 - 2\lambda + 2 \qquad \lambda_{1,2} = 1 \pm i$$

Eigenvectors are resp. $\mathbf{v}_1 = (1, i)^T$, $\mathbf{v}_2 = (1, -i)^T$. By Theorem 6.5, if $A_{n\times n}$ has $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ with resp. eigenvectors $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$, $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$. Then two real $\mathbf{x_1}(t) = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t)$, $\mathbf{x_2}(t) = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t)$. Hence here $n = 2$, $\alpha = 1 = \beta$, $\mathbf{a} = (1, 0)^T$, $\mathbf{b} = (0, 1)^T$. We have

$$\mathbf{x_1}(t) = e^t(\begin{pmatrix}1\\0\end{pmatrix}\cos t - \begin{pmatrix}0\\1\end{pmatrix}\sin t)$$

$$\mathbf{x_2}(t) = e^t(\begin{pmatrix}1\\0\end{pmatrix}\sin t + \begin{pmatrix}0\\1\end{pmatrix}\cos t)$$

$(x_1(t), x_2(t)) = (0, 0)^T$ is an equilibrium solution. Since $\alpha > 0$ (in the case of complex conjugate roots), the point $(0, 0)$ in the phase-portrait plane is unstable spiral point (as $t \to \infty$)
27 (a) Substitute $y = t$ to both sides of

$$t^2y'' + ty' - y = 0$$

to check if the equation holds. Do the same for $y = 1/t$.
(b) Substitute $y(t) = 2t^2$ to the inhomogeneous equation (*) to check if the equation holds. If so that will verify it is a solution.
(c) The general solution $y_{gen} = y_h + y_p = C_1t + C_2/t + 2t^2$
(d) Plugging in the initial data at $t = 1$ to y_{gen} to determine the constants C_i.
(e) Divided by t^2 on both sides of $(*)$, then the equation becomes

$$y'' + \frac{1}{t}y' - \frac{1}{t^2}y = 6$$

Apply the E and U theorem, since the functions $p(t) = 1/t, q(t) = -1/t$ and $f(t) = 6$ are all continuous on $(0, \infty)$ and $(-\infty, 0)$, we obtain that the equation (*) is guaranteed to have a unique solution on either of these two intervals. Further, since the solution of $(**)$ $y(t)$ is required to satisfy the I.C. at $t = 1$, we must conclude that the largest interval is $(0, \infty)$. This answer is consistent with (c) and (d). So the E and U theorem is sharp, in other words, in general we do not expect the interval of existence to be larger than where $p(t), q(t), f(t)$ are all continuous.