Review Exam
Math 3230

Name
Id
Section

Read each problem carefully. Avoid simple mistakes. To receive full credits you must show your work to support your answer. An incomplete answer might receive partial credits if you have written down a reasonable solution.

(1) Verify that the equation $y^2 = x^2 + C$, where C is a constant, satisfies the differential equation
$$\frac{dy}{dx} = \frac{x}{y}$$

(2) Find the exact solution of the initial value problem
$$\frac{dy}{dx} = e^{x-y}, \quad y(0) = -1$$

(3) Find the general solution to 1^{st}-order linear ODE
$$x' - x \sin t = 2te^{-\cos t}$$

(4) * Find a general solution of the Bernoulli equation
$$\frac{dy}{dx} = 6\frac{y}{x} - xy^2$$
(Hint: Substitution $w = y^{1-n}$ (n=2) to convert it to a first-order linear equation involving $w = w(x)$)

(5) Figure 1 (Page 7) is the direction field for $y' = y(y-1)(y+1)$.
  (a) Draw on the direction field the solutions of the differential equation satisfying each of the following initial values.
    (i) $y(0) = 0.0$
    (ii) $y(0) = 0.5$
    (iii) $y(0) = -1.5$
  (b) For the solution $y(t)$ with initial condition $y(0) = 0.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?
  (c) For the solution $y(t)$ with initial condition $y(0) = -1.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?
  (d) The constant solutions $y = 0$, $y = 1$, $y = -1$ are three equilibrium solutions. Which one is asymptotically stable, unstable or semi-stable?

(6) a) Find a fundamental set of solutions of the homogeneous $y'' - 6y' + 9y = 0$
b) Explain why the two solutions you find in part (a) are linearly independent.
c) Use the method of undetermined coefficients to find the general solution of
$$y'' - 6y' + 9y = e^{3t}.$$
(Hint: In (c), since e^{3t} and te^{3t} are solutions of the homogeneous equation with repeated root, the particular solution should be t^2e^{3t}, according to the Remark on 2nd order linear ODE, chapter 4)

(7) Use variation of parameters to find the general solution to the equation

$$x'' + x = \csc t$$

(Hint: Let $x(t) = u_1 f_1 + u_2 f_2$, where f_1, f_2 are fundamental solutions of $x'' + x = 0$. Then $u_1'(t) = -g(t)f_2/W$, $u_2'(t) = g(t)f_1/W$, where $g = \csc t = 1/\sin t$ and $W = W(f_1, f_2)$ is the Wronskian of f_1, f_2.)

(8) Answer the following questions concerning the differential equation

$$t^2 y'' + ty' - y = 6t^2. \tag{$*$}$$

(a) Verify that $\varphi_1(t) = t$ and $\varphi_2(t) = \dfrac{1}{t}$ are solutions of the associated homogeneous differential equation $t^2 y'' + ty' - y = 0$.
(b) Verify that $y_p(t) = 2t^2$ is a solution to equation $(*)$.
(c) What is the general solution of equation $(*)$?
(d) What is the solution of the initial value problem

$$t^2 y'' + ty' - y = 6t^2, \quad y(1) = -1, \quad y'(1) = 1? \tag{$**$}$$

(e) What is the largest interval on which the initial value problem $(**)$ is guaranteed to have a solution by the existence and uniqueness theorem? Is this answer consistent with the solution that you found in the previous part of this exercise?

(9) * Write the following initial value problem as a system of first-order, linear differential equations in matrix notation

$$y'' + 2y' + 4y = 3\cos 2t, \qquad y(0) = 1, y'(0) = 0$$

(10) a) Given an example of a set of three functions so that they are Linear Independent.
b) Given an example of a set of three functions so that they are Linear Dependent.
c) Suppose $p(t), q(t)$ and $f(t)$ are ________on an open interval (a, b) containing $t = t_0$. Then the IVP

$$y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = y_0, y'(t_0) = y_1$$

has a ________ solution on (a, b)
d) Use the Existence and Uniqueness Theorem for linear IVPs to determine the largest interval on which the solution is guaranteed to exist.

$$y'' + \frac{y'}{x-5} + \frac{y}{(x-5)^2} = \frac{\sin x}{2x+3}, \; y(0) = \pi, \; y'(0) = 0.5\pi$$

(11) Determine whether $\Phi(t) = \begin{pmatrix} \cos t + 2\sin t & 2\cos t - \sin t \\ \sin t & \cos t \end{pmatrix}$ is a fundamental matrix for the linear system $X'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X(t)$. If so, give an expression of a general solution.

(12) a) Verify $\Phi(t) = \begin{pmatrix} 3e^{4t} & e^{3t} \\ 2e^{4t} & e^{3t} \end{pmatrix}$ is a fundamental matrix for the system $X'(t) = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} X(t)$
b) Find a general solution to the system in (a)

(13) * (repeated eigenvalues) Find the general solution to the system

$$Y' = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} Y$$

(14) The eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are $\lambda = 1, 2, 3$ and $(0,0,1)^T$, $(1,2,1)^T$, and $(1,3,0)^T$, respectively. Consider the IVP

$$X' = AX, \quad X(0) = X_0$$

where $X_0 = (3,7,4)^T$. a) Find a fundamental set of solutions to the 3 by 3 linear system $X' = AX$, and verify the linear independency using the Wronskian. b) Solve the IVP.

(15) Use the method of undetermined coefficients to find a general solution for

$$Y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y + \begin{pmatrix} t \\ -1 \end{pmatrix}$$

(16) * Use variation of parameters to find a general solution for

$$X' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

(17) * Consider the system

$$Y' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} Y \qquad -\infty < t < \infty$$

(a) Find a real-valued fundamental set of solutions of general solution. (b) Identify its equilibrium solution, classify the type and stability characteristics of the equilibrium point.

Answer

1 Take differential of $y^2 = x^2 + C$ both sides in x variable. We have

$$\begin{aligned} d(y^2) &= d(x^2 + C) \\ 2ydy &= 2xdx \\ \frac{dy}{dx} &= x/y. \end{aligned}$$

2

$$\frac{dy}{dx} = e^{x-y}, \quad y(0) = -1$$

Separating variable and then integrating both sides, we get

$$\begin{aligned} & e^y dy = e^x dx \\ & \int e^y dy = \int e^x dx \\ & e^y = e^x + C \; or \; y = \ln(e^x + C) \end{aligned}$$

Substitute $y(0) = -1$ to get $e^{-1} = e^0 + C$, hence $C = 1/e - 1$.

3 The standard equation reads $x' + P(t)x = Q(t)$. $P = -\sin t$, $Q = 2te^{-\cos t}$. Use integrating factor method or the formula

$$x(t) = e^{-\int P}\left(\int e^{\int P} Q dt\right)$$

5 An autonomous differential equations are of the form $\frac{d}{dt} = f(u)$ where $f = f(u)$ is a function depending on u only. Thus the zero(s) of f are called equilibrium solutions or equilibrium points.

Classify the (constant) solutions. If solutions start near an equilibrium solution will they move away from the equilibrium solution or towards the equilibrium solution?

Equilibrium solutions in which solutions that start near them move away from the equilibrium solution are called **unstable equilibrium points** or unstable equilibrium solutions.

Equilibrium solutions in which solutions that start near them move toward the equilibrium solution are called **asymptotically stable equilibrium points** or asymptotically stable equilibrium solutions.

The equilibrium solution is called **semi-stable** provided solutions on one side of an equilibrium solution move towards the equilibrium solution and on the other side of the equilibrium solution move away from it.

6 (c) Undetermined coefficients method.

$$y'' - 6y' + 9y = e^{3t}.$$

Since e^{3t} and te^{3t} are solutions of the homogeneous equation with repeated root, the particular solution should be Ct^2e^{3t}. To determine the constant C, plug in the inhomogeneous ODE:

$$\begin{aligned}
&(Ct^2e^{3t})' = Ce^{3t}(2t + 3t^2) \\
&(Ct^2e^{3t})'' = Ce^{3t}(2 + 12t + 9t^2) \\
&Ce^{3t}(2 + 12t + 9t^2) - 6(Ce^{3t}(2t + 3t^2)) + 9(Ce^{3t}t^2) = e^{3t} \\
&2C = 1 \Rightarrow C = 1/2 \\
&Hence \quad y_p = \frac{1}{2}e^{3t^2}.
\end{aligned}$$

10 (d) The Existence and Uniqueness Theorem for second order linear equation states that if the coefficients are continuous on a common interval (a, b), then the ODE has a unique solution $y = y(t)$ defined on (a, b) that satisfies the second order ODE. Now $p(t) = 1/(x-5), q(t) = 1/(x-5)2, f(t) = \sin x/(2x+3)$, where the largest interval containing $x = 0$ on which all p, q, f are continuous is $(-3/2, 5)$. So this is the largest interval on which the solution is guaranteed to exist for the second order linear initial value problem.

11 Define Φ_1 to be the first column of Φ, and Φ_1 to be the second column of Φ. Calculate the Wronski of $\Phi(t)$ to find that $W[\Phi_1, \Phi_2] = \det\begin{pmatrix} \cos t + 2\sin t & 2\cos t - \sin t \\ \sin t & \cos t \end{pmatrix} = \cos^2 t +$

$\sin^2 t = 1 \neq 0$. This means Φ_1, Φ_2 are linear independent. So the general solution is given by

$$X(t) = \Phi(t)\mathbf{C} = c_1\Phi_1(t) + c_2\Phi_2(t).$$

13 Step 1. Solve $|\lambda - A| = 0$

$$\begin{vmatrix} \lambda+1 & 1 \\ -1 & \lambda+3 \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2,$$

to get $\lambda_{1,2} = -2$.

Step 2. Solve $(\lambda - A)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ for $\lambda = -2$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Longrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step 3. Since this system has double eigenvalue. We need to find the other eigenvector Solve for $\lambda = -2$

$$(A - \lambda)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \Longrightarrow \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(Here any solution satisfying $w_1 - w_2 = 1$ will be correct)

Step 4. Now we construct the two linear independent solutions

$$e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, e^{\lambda t}(t\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + w)$$

that is

$$e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, e^{-2t}(t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix})$$

Therefore the general solution is

$$C_1\begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} + C_2\begin{pmatrix} (t+1)e^{-2t} \\ te^{-2t} \end{pmatrix}.$$

14 (a) The fundamental matrix is

$$\Phi(t) = \left(e^t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right)$$

Since the determinant $|\Phi(t)| = e^{6t} \neq 0$, the three columns are linear independent.

(b) The general solution is given by

$$X(t) = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 e^t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

Plugging the initial condition at $t = 0$ we have

$$\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} c_2 + c_3 \\ 2c_2 + 3c_3 \\ c_1 + c_2 \end{pmatrix}$$

Solve this linear system to obtain $c_1 = 2, c_2 = 2, c_3 = 1$.

17* Step 1. Find eigenvalues, $|\lambda - A| = 0$. Solve

$$\begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1,$$

to get $\lambda_1 = 1 + i = \alpha + i\beta$, $\lambda_2 = 1 - i = \alpha - i\beta$, where $\alpha = 1, \beta = 1$.

Step 2. Find eigenvectors. $(\lambda - A)\mathbf{v} = \mathbf{0}$ Solve $(\lambda - A)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ for $\lambda = 1 + i$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} := \mathbf{a} + i\mathbf{b}$$

where $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Step 3. Construct a pair of real-valued solutions. According to Section 6.4, we have

$$X_1(t) = e^{\alpha t}(\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t)$$
$$X_2(t) = e^{\alpha t}(\mathbf{b} \cos \beta t + \mathbf{a} \sin \beta t)$$

FIGURE 1. Direction Field for Problem 5