10

Applications

51. Vibration of a spring

Consider a steel spring attached to a support and hanging downward. Within certain elastic limits the spring will obey Hooke's law: if the spring is stretched or compressed, its change in length will be proportional to the force exerted upon it and, when that force is removed, the spring will return to its original position with its length and other physical properties unchanged. There is, therefore, associated with each spring a numerical constant, the ratio of the force exerted to the displacement produced by that force. If a force of magnitude Q pounds (lb) stretches the spring c feet (ft), the relation

$$Q = kc \tag{1}$$

defines the spring constant k in units of pounds per foot (lb/ft).

Let a body B weighing w lb be attached to the lower end of a spring (Figure 13) and brought to the point of equilibrium where it can remain at rest. Once the weight B is moved from the point of equilibrium E in Figure 14,

the motion of B will be determined by a differential equation and associated initial conditions.

Let t be time measured in seconds after some initial moment when the motion begins. Let x, in feet, be distance measured positive downward (negative upward) from the point of equilibrium, as in Figure 14. We assume that the motion of B takes place entirely in a vertical line, so the velocity and acceleration are given by the first and second derivatives of x with respect to t.

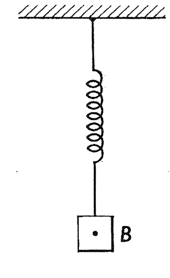

FIGURE 13

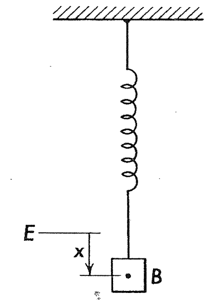

FIGURE 14

In addition to the force proportional to displacement (Hooke's law), there will in general be a retarding force caused by resistance of the medium in which the motion takes place or by friction. We are interested here only in such retarding forces as can be well approximated by a term proportional to the velocity because we restrict our study to problems involving linear differential equations. Such a retarding force will contribute to the total force acting on B a term $bx'(t)$, in which b is a constant to be determined experimentally for the medium in which the motion takes place. Some common retarding forces, such as one proportional to the cube of the velocity, lead to nonlinear differential equations.

The weight of the spring is usually negligible compared to the weight B, so we use for the mass of our system the weight of B divided by g, the constant acceleration of gravity. If no forces other than those described above act upon the weight, the displacement x must satisfy the equation

$$\frac{w}{g}x''(t) + bx'(t) + kx(t) = 0. \tag{2}$$

Suppose that an additional vertical force, due to the motion of the support or to presence of a magnetic field, and so on, is imposed upon the system. The new, impressed force, will depend upon time and we may use $F(t)$ to denote the acceleration that it alone would impart to the weight B. Then the impressed force is $(w/g)F(t)$ and equation (2) is replaced by

$$\frac{w}{g}x''(t) + bx'(t) + kx(t) = \frac{w}{g}F(t). \qquad (3)$$

At time zero, let the weight be displaced by an amount x_0 from the equilibrium point and let the weight be given an initial velocity v_0. Either or both of x_0 and v_0 may be zero in specific instances. The problem of determining the position of the weight at any time t becomes that of solving the initial value problem consisting of the differential equation

$$\frac{w}{g}x''(t) + bx'(t) + kx(t) = \frac{w}{g}F(t), \qquad \text{for } t > 0, \qquad (4)$$

and the initial conditions

$$x(0) = x_0, \qquad x'(0) = v_0. \qquad (5)$$

It is convenient to rewrite equation (4) in the form

$$x''(t) + 2\gamma x'(t) + \beta^2 x(t) = F(t), \qquad (6)$$

in which we have put

$$\frac{bg}{w} = 2\gamma, \qquad \frac{kg}{w} = \beta^2.$$

We may choose $\beta > 0$ and we know $\gamma \geqq 0$. Note that $\gamma = 0$ corresponds to a negligible retarding force.

A number of special cases of the initial value problem contained in equations (5) and (6) will now be studied.

52. Undamped vibrations

If $\gamma = 0$ in the problem of Section 51, the differential equation becomes

$$x''(t) + \beta^2 x(t) = F(t), \qquad (1)$$

a second-order linear equation with constant coefficients in which $\beta^2 = kg/w$. The complementary function associated with the homogeneous equation $x''(t) + \beta^2 x(t) = 0$ is

$$x_c = c_1 \sin \beta t + c_2 \cos \beta t,$$

and the general solution of equation (1) will be of the form

$$x = c_1 \sin \beta t + c_2 \cos \beta t + x_p, \qquad (2)$$

where x_p is any particular solution of the nonhomogeneous equation.

We now look at a number of examples of the motion described by equation (2) for different functions $F(t)$ in equation (1).

EXAMPLE (a): Solve the spring problem with no damping but with $F(t) = A \sin \omega t$, where $\beta \neq \omega$. The case $\beta = \omega$ leads to resonance, which will be discussed in the next section.

The differential equation of motion is

$$\frac{w}{g}x''(t) + kx(t) = \frac{w}{g}A \sin \omega t$$

and may be written

$$x''(t) + \beta^2 x(t) = A \sin \omega t, \qquad (3)$$

with the introduction of $\beta^2 = kg/w$. We shall assume initial conditions

$$x(0) = x_0, \qquad x'(0) = v_0. \qquad (4)$$

A particular solution of equation (3) will be of the form

$$x_p = E \sin \omega t,$$

and we may obtain E by direct substitution into equation (3). We have

$$-E\omega^2 \sin \omega t + \beta^2 E \sin \omega t = A \sin \omega t,$$

an equation that is satisfied for all t only if we choose

$$E = \frac{A}{\beta^2 - \omega^2}.$$

The general solution of (3) now becomes

$$x(t) = c_1 \sin \beta t + c_2 \cos \beta t + \frac{A}{\beta^2 - \omega^2} \sin \omega t \qquad (5)$$

with derivative

$$x'(t) = c_1 \beta \cos \beta t - c_2 \beta \sin \beta t + \frac{A\omega}{\beta^2 - \omega^2} \cos \omega t.$$

The initial conditions (4) now require

$$x_0 = c_2 \qquad \text{and} \qquad v_0 = c_1 \beta + \frac{A\omega}{\beta^2 - \omega^2}$$

and force us to choose

$$c_1 = \frac{v_0}{\beta} - \frac{A\omega}{\beta(\beta^2 - \omega^2)} \qquad \text{and} \qquad c_2 = x_0.$$

From (5) it follows at once that

$$x(t) = \frac{v_0}{\beta}\sin\beta t + x_0\cos\beta t - \frac{A\omega}{\beta(\beta^2-\omega^2)}\sin\beta t + \frac{A}{\beta^2-\omega^2}\sin\omega t. \quad (6)$$

The x of (6) has two parts. The first two terms represent the natural simple harmonic component of the motion, a motion that would be present if A were zero. The last two terms in (6) are caused by the presence of the external force $(w/g)A\sin\omega t$.

EXAMPLE (b): A spring is such that it would be stretched 6 inches (in.) by a 12-lb weight. Let the weight be attached to the spring and pulled down 4 in. below the equilibrium point. If the weight is started with an upward velocity of 2 ft/sec, describe the motion. No damping or impressed force is present.

We know that the acceleration of gravity enters our work in the expression for the mass. We wish to use the value $g = 32$ feet per second per second (ft/sec^2) and we must use consistent units, so we put all lengths into feet.

First we determine the spring constant k from the fact that the 12-lb weight stretches the spring 6 in., $\frac{1}{2}$ ft. Thus $12 = \frac{1}{2}k$ so that $k = 24$ lb/ft.

The differential equation of the motion is therefore

$$\tfrac{12}{32}x''(t) + 24x(t) = 0. \quad (7)$$

At time zero the weight is 4 in. ($\frac{1}{3}$ ft) below the equilibrium point, so $x(0) = \frac{1}{3}$. The initial velocity is negative (upward), so $x'(0) = -2$. Thus our problem is that of solving

$$x''(t) + 64x(t) = 0; \qquad x(0) = \tfrac{1}{3},\ x'(0) = -2. \quad (8)$$

The general solution of equation (8) is

$$x(t) = c_1\sin 8t + c_2\cos 8t,$$

from which

$$x'(t) = 8c_1\cos 8t - 8c_2\sin 8t.$$

The initial conditions now require that

$$\tfrac{1}{3} = c_2 \qquad \text{and} \qquad -2 = 8c_1,$$

so that finally

$$x(t) = -\tfrac{1}{4}\sin 8t + \tfrac{1}{3}\cos 8t. \quad (9)$$

A detailed study of the motion is straightforward once (9) has been obtained. The amplitude of the motion is

$$\sqrt{(\tfrac{1}{3})^2 + (\tfrac{1}{4})^2} = \tfrac{5}{12};$$

that is, the weight oscillates between points 5 in. above and below E. The period is $\frac{1}{4}\pi$ sec.

53. Resonance

In Example (a) of the previous section we postponed the study of the special case, $\beta = \omega$. In that case, the differential equation to be solved is

$$x''(t) + \beta^2 x(t) = A\sin\beta t, \quad (1)$$

where we had let $\beta^2 = kg/w$.

The complementary function associated with the homogeneous equation $x''(t) + \beta^2x(t) = 0$ will be the same as it was before, but the previous particular solution x_p will not exist because $\beta = \omega$.

The method of undetermined coefficients may be applied here to seek a particular solution of the form

$$x_p = Pt\sin\beta t + Qt\cos\beta t, \quad (2)$$

where P and Q are constants to be determined. Direct substitution of the x_p of (2) into equation (1) yields

$$2P\beta\cos\beta t - 2Q\beta\sin\beta t = A\sin\beta t,$$

an equation that can be satisfied for all t only if $P = 0$ and $Q = -A/2\beta$. Thus

$$x_p = \frac{-At}{2\beta}\cos\beta t, \quad (3)$$

and the general solution of (1) is

$$x(t) = c_1\sin\beta t + c_2\cos\beta t - \frac{At}{2\beta}\cos\beta t, \quad (4)$$

from which we obtain

$$x'(t) = c_1\beta\cos\beta t - c_2\beta\sin\beta t + \frac{At}{2}\sin\beta t - \frac{A}{2\beta}\cos\beta t.$$

The initial conditions $x(0) = x_0$ and $x'(0) = v_0$ now force us to take

$$c_2 = x_0 \qquad \text{and} \qquad c_1 = \frac{v_0}{\beta} + \frac{A}{2\beta^2}.$$

The final solution may now be written

$$x(t) = x_0\cos\beta t + \frac{v_0}{\beta}\sin\beta t + \frac{A}{2\beta^2}(\sin\beta t - \beta t\cos\beta t). \quad (5)$$

That (5) satisfies the initial value problem is readily verified.

In the solution (5) the terms proportional to $\cos \beta t$ and $\sin \beta t$ are bounded, but the term with $\beta t \cos \beta t$ can be made as large as we wish by proper choice of t. This building up of large amplitudes in the vibration is called *resonance*.

Exercises

1. A spring is such that a 5-lb weight stretches it 6 in. The 5-lb weight is attached, the spring reaches equilibrium, then the weight is pulled down 3 in. below the equilibrium point and started off with an upward velocity of 6 ft/sec. Find an equation giving the position of the weight at all subsequent times.
ANS. $x = \frac{1}{4}(\cos 8t - 3 \sin 8t)$.

2. A spring is stretched 1.5 in. by a 2-lb weight. Let the weight be pushed up 3 in. above E and then released. Describe the motion. ANS. $x = -\frac{1}{4}\cos 16t$.

3. For the spring and weight of exercise 2, let the weight be pulled down 4 in. below E and given a downward initial velocity of 8 ft/sec. Describe the motion.
ANS. $x = \frac{1}{3}\cos 16t + \frac{1}{2}\sin 16t$.

4. Show that the answer to exercise 3 can be written $x = 0.60 \sin (16t + \phi)$ where $\phi = \arctan \frac{2}{3}$.

5. A spring is such that a 4-lb weight stretches it 6 in. An impressed force $\frac{1}{2}\cos 8t$ is acting on the spring. If the 4-lb weight is started from the equilibrium point with an imparted upward velocity of 4 ft/sec, determine the position of the weight as a function of time. ANS. $x = \frac{1}{4}(t - 2) \sin 8t$.

6. A spring is such that it is stretched 6 in. by a 12-lb weight. The 12-lb weight is pulled down 3 in. below the equilibrium point and then released. If there is an impressed force of magnitude $9 \sin 4t$ lb, describe the motion. Assume that the impressed force acts downward for very small t.
ANS. $x = \frac{1}{4}\cos 8t - \frac{1}{4}\sin 8t + \frac{1}{2}\sin 4t$.

7. Show that the answer to exercise 6 can be written

$$x = \tfrac{1}{4}\sqrt{2}\cos (8t + \pi/4) + \tfrac{1}{2}\sin 4t.$$

8. A spring is such that a 2-lb weight stretches it $\frac{1}{2}$ ft. An impressed force $\frac{1}{4}\sin 8t$ is acting upon the spring. If the 2-lb weight is released from a point 3 in. below the equilibrium point, determine the equation of motion.
ANS. $x = \frac{1}{4}(1 - t)\cos 8t + \frac{1}{32}\sin 8t$ (ft).

9. For the motion of exercise 8, find the first four times at which stops occur and find the position at each stop. ANS. $t = \pi/8, \pi/4, 1, 3\pi/8$ (sec)
and $x = -0.15, +0.05, +0.03, +0.04$ (ft), respectively.

10. Determine the approximate position to be expected, if nothing such as breakage interferes, at the time of the 65th stop, when $t = 8\pi$ (sec), in exercise 8.
ANS. $x = -6.0$ (ft).

11. A spring is such that a 16-lb weight stretches it 1.5 in. The weight is pulled down to a point 4 in. below the equilibrium point and given an initial downward velocity of 4 ft/sec. An impressed force of $360 \cos 4t$ lb is applied. Find the position and velocity of the weight at time $t = \pi/8$ sec.
ANS. At $t = \pi/8$ (sec), $x = -\frac{8}{3}$ (ft), $v = -8$ (ft/sec).

12. A spring is stretched 3 in. by a 5-lb weight. Let the weight be started from E with an upward velocity of 12 ft/sec. Describe the motion. ANS. $x = -1.06 \sin 11.3t$

13. For the spring and weight of exercise 12, let the weight be pulled down 4 in. below E and then given an upward velocity of 8 ft/sec. Describe the motion.
ANS. $x = 0.33 \cos 11.3t - 0.71 \sin 11.3t$.

14. Find the amplitude of the motion in exercise 13. ANS. 0.78 ft.

15. A 20-lb weight stretches a certain spring 10 in. Let the spring first be compressed 4 in., and then the 20-lb weight attached and given an initial downward velocity of 8 ft/sec. Find how far the weight would drop. ANS. 35 in.

16. A spring is such that an 8-lb weight would stretch it 6 in. Let a 4-lb weight be attached to the spring, which is then pushed up 2 in. above its equilibrium point and released. Describe the motion. ANS. $x = -\frac{1}{6}\cos 11.3t$.

17. If the 4-lb weight of exercise 16 starts at the same point, 2 in. above E, but with an upward velocity of 15 ft/sec, when will the weight reach its lowest point?
ANS. At t = approximately 0.4 sec.

18. A spring is such that it is stretched 4 in. by a 10-lb weight. Suppose the 10-lb weight to be pulled down 5 in. below E and then given a downward velocity of 15 ft/sec. Describe the motion.
ANS. $x = 0.42 \cos 9.8t + 1.53 \sin 9.8t$
$= 1.59 \cos (9.8t - \phi)$, where $\phi = \arctan 3.64$.

19. A spring is such that it is stretched 4 in. by an 8-lb weight. Suppose the weight to be pulled down 6 in. below E and then given an upward velocity of 8 ft/sec. Describe the motion. ANS. $x = 0.50 \cos 9.8t - 0.82 \sin 9.8t$.

20. Show that the answer to exercise 19 can be written $x = 0.96 \cos (9.8t + \phi)$ where $\phi = \arctan 1.64$.

21. A spring is such that a 4 lb weight stretches it 6 in. The 4-lb weight is attached to the vertical spring and reaches its equilibrium point. The weight is then ($t = 0$) drawn downward 3 in. and released. There is a simple harmonic exterior force equal to $\sin 8t$ impressed upon the whole system. Find the time for each of the first four stops following $t = 0$. Put the stops in chronological order.
ANS. $t = \pi/8, \frac{1}{2}, \pi/4, 3\pi/8$ (sec).

22. A spring is stretched 1.5 in. by a 4-lb weight. Let the weight be pulled down 3 in. below equilibrium and released. If there is an impressed force $8 \sin 16t$ acting upon the spring, describe the motion. ANS. $x = \frac{1}{4}(1 - 8t)\cos 16t + \frac{1}{8}\sin 16t$.

23. For the motion of exercise 22, find the first four times at which stops occur and find the position at each stop. ANS. $t = \frac{1}{8}, \pi/16, \pi/8, 3\pi/16$ (sec) and
$x = +0.11, +0.14, -0.54, +0.93$ (ft), respectively.

54. Damped vibrations

In the general linear spring problem of Section 51, we were confronted with

$$x''(t) + 2\gamma x'(t) + \beta^2 x(t) = F(t); \qquad x(0) = x_0,\ x'(0) = v_0, \tag{1}$$

in which $2\gamma = bg/w$ and $\beta^2 = kg/w$, $\beta > 0$. The auxiliary equation $m^2 + 2\gamma m + \beta^2 = 0$ has roots $-\gamma \pm \sqrt{\gamma^2 - \beta^2}$ and we see that the nature of the complementary function depends upon whether $\beta > \gamma$, $\beta = \gamma$, or $\beta < \gamma$.

If $\beta > \gamma$, $\beta^2 - \gamma^2 > 0$, so let us put

$$\beta^2 - \gamma^2 = \delta^2. \tag{2}$$

Then the general solution of (1) will be

$$x(t) = e^{-\gamma t}(c_1 \cos \delta t + c_2 \sin \delta t) + \psi_1(t), \tag{3}$$

in which $\psi_1(t)$ is any particular solution of equation (1). The presence of the function $e^{-\gamma t}$, called a damping factor, will cause the natural part of the solution, that is, the part independent of the external force $(w/g)F(t)$, to approach zero as $t \to \infty$.

If in (1) we have $\beta = \gamma$, the two roots of the auxiliary equation are equal and the general solution becomes

$$x(t) = e^{-\gamma t}(c_1 + c_2 t) + \psi_2(t), \tag{4}$$

in which $\psi_2(t)$ is a particular solution of (1). Again the natural component has the damping factor $e^{-\gamma t}$ in it.

If in (1) we have $\beta < \gamma$ and $\gamma^2 - \beta^2 > 0$, then we can set

$$\gamma^2 - \beta^2 = \sigma^2, \qquad \sigma > 0. \tag{5}$$

Since $\sigma < \gamma$, the two roots of the auxiliary equation are both real and negative, and we have

$$x(t) = c_1 e^{(-\gamma+\sigma)t} + c_2 e^{(-\gamma-\sigma)t} + \psi_3(t). \tag{6}$$

Again $\psi_3(t)$ is a particular solution of (1), and we see that the damping factor $e^{-\gamma t}$ causes the natural component of (6) to approach zero as $t \to \infty$.

Suppose for the moment that we have $F(t) \equiv 0$, so the natural component of the motion is all that is under consideration. If $\beta > \gamma$, equation (3) holds and the motion is a *damped oscillatory* one. If $\beta = \gamma$, equation (4) holds and the motion is not oscillatory; it is called *critically damped* motion. If $\beta < \gamma$, (6) holds and the motion is said to be *overdamped*; the parameter γ is larger than it needs to be to remove the oscillations. Figure 15 shows a representative graph of each type of motion mentioned in this paragraph, a damped oscillatory motion, a critically damped motion, and an overdamped motion.

EXAMPLE: Solve the problem of Example (b), Section 52, with an added damping force of magnitude $0.6|v|$. Such a damping force can be realized by immersing the weight B in a thick liquid.

The initial value problem to be solved is

$$\tfrac{12}{32}x''(t) + 0.6x'(t) + 24x(t) = 0; \qquad x(0) = \tfrac{1}{3},\ x'(0) = -2. \tag{7}$$

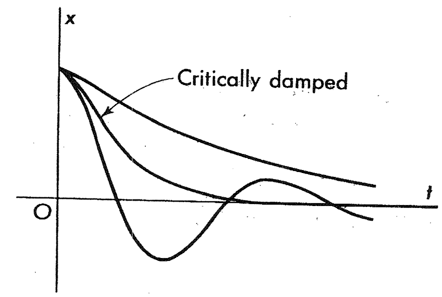

FIGURE 15

The auxiliary equation of (7) may be written

$$m^2 + 1.6m + 64 = 0,$$

an equation that has roots $-0.8 \pm \sqrt{63.36}\,i$. Therefore, the general solution of (7) is

$$x(t) = e^{-0.8t}(c_1 \cos 8.0t + c_2 \sin 8.0t)$$

and

$$x'(t) = e^{-0.8t}[(-8c_1 - 0.8c_2)\sin 8.0t + (8c_2 - 0.8c_1)\cos 8.0t].$$

The initial conditions in (7) now give us

$$\tfrac{1}{3} = c_1 \qquad \text{and} \qquad -2 = 8c_2 - 0.8c_1,$$

so that $c_1 = 0.33$ and $c_2 = -0.22$.

Therefore the desired solution is

$$x(t) = \exp(-0.8t)(0.33 \cos 8.0t - 0.22 \sin 8.0t), \tag{8}$$

a portion of its graph being shown in Figure 16.

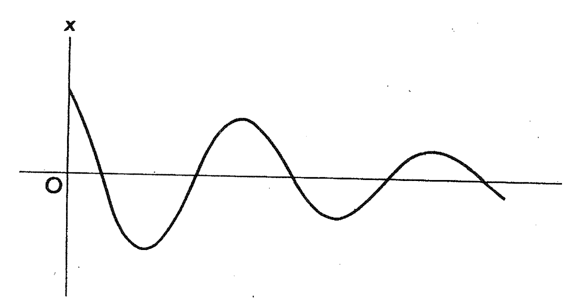

FIGURE 16

Exercises

1. A certain straight-line motion is determined by the differential equation

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + 169x = 0$$

and the conditions that when $t = 0$, $x = 0$, and $v = 8$ ft/sec.

(a) Find the value of γ that leads to critical damping, determine x in terms of t, and draw the graph for $0 \leq t \leq 0.2$. ANS. $\gamma = 13(1/\text{sec})$, $x = 8t\,e^{-13t}$.

(b) Use $\gamma = 12$. Find x in terms of t and draw the graph. ANS. $x = 1.6\,e^{-12t}\sin 5t$.

(c) Use $\gamma = 14$. Find x in terms of t and draw the graph. ANS. $x = 0.77(e^{-8.8t} - e^{-19.2t})$.

2. A spring is such that a 2-lb weight stretches it $\frac{1}{2}$ ft. An impressed force $\frac{1}{4}\sin 8t$ and a damping force of magnitude $|v|$ are both acting on the spring. The weight starts $\frac{1}{4}$ ft below the equilibrium point with an imparted upward velocity of 3 ft/sec. Find a formula for the position of the weight at time t. ANS. $x = \frac{3}{32}e^{-8t}(3 - 8t) - \frac{1}{32}\cos 8t$.

3. A spring is such that a 4-lb weight stretches it 0.64 ft. The 4-lb weight is pushed up $\frac{1}{3}$ ft above the point of equilibrium and then started with a downward velocity of 5 ft/sec. The motion takes place in a medium which furnishes a damping force of magnitude $\frac{1}{4}|v|$ at all times. Find the equation describing the position of the weight at time t. ANS. $x = \frac{1}{3}e^{-t}(2\sin 7t - \cos 7t)$.

4. A spring is such that a 4-lb weight stretches it 0.32 ft. The weight is attached to the spring and moves in a medium which furnishes a damping force of magnitude $\frac{3}{2}|v|$. The weight is drawn down $\frac{1}{2}$ ft below the equilibrium point and given an initial upward velocity of 4 ft/sec. Find the position of the weight thereafter. ANS. $x = \frac{1}{8}e^{-6t}(4\cos 8t - \sin 8t)$.

5. A spring is such that a 4-lb weight stretches the spring 0.4 ft. The 4-lb weight is attached to the spring (suspended from a fixed support) and the system is allowed to reach equilibrium. Then the weight is started from equilibrium position with an imparted upward velocity of 2 ft/sec. Assume that the motion takes place in a medium that furnishes a retarding force of magnitude numerically equal to the speed, in feet per second, of the moving weight. Determine the position of the weight as a function of time. ANS. $x = -\frac{1}{4}e^{-4t}\sin 8t$.

6. A spring is stretched 6 in. by a 3-lb weight. The 3-lb weight is attached to the spring and then started from equilibrium with an imparted upward velocity of 12 ft/sec. Air resistance furnishes a retarding force equal in magnitude to $0.03|v|$. Find the equation of motion. ANS. $x = -1.5\,e^{-0.16t}\sin 8t$.

7. A spring is such that a 2-lb weight stretches it 6 in. There is a damping force present, with magnitude the same as the magnitude of the velocity. An impressed force $(2\sin 8t)$ is acting on the spring. If, at $t = 0$, the weight is released from a point 3 in. below the equilibrium point, find its position for $t > 0$. ANS. $x = (\frac{1}{2} + 4t)\,e^{-8t} - \frac{1}{4}\cos 8t$.

8. A spring is stretched 10 in. by a 4-lb weight. The weight is started 6 in. below the equilibrium point with an upward velocity of 8 ft/sec. If a resisting medium furnishes a retarding force of magnitude $\frac{1}{4}|v|$, describe the motion. ANS. $x = e^{-t}[0.50\cos 6.1t - 1.23\sin 6.1t]$.

9. For exercise 8, find the times of the first three stops and the position (to the nearest inch) of the weight at each stop. ANS. $t_1 = 0.3$ sec, $x_1 = -12$ in.; $t_2 = 0.8$ sec, $x_2 = +6$ in.; $t_3 = 1.3$ sec, $x_3 = -4$ in.

10. A spring is stretched 4 in. by a 2-lb weight. The 2-lb weight is started from the equilibrium point with a downward velocity of 12 ft/sec. If air resistance furnishes a retarding force of magnitude 0.02 of the velocity, describe the motion. ANS. $x = 1.22\,e^{-0.16t}\sin 9.8t$.

11. For exercise 10, find how long it takes the damping factor to drop to one-tenth its initial value. ANS. 14.4 sec.

12. For exercise 10, find the position of the weight at: (a) the first stop; (b) the second stop. ANS. (a) $x = 1.2$ ft; (b) $x = -1.1$ ft.

13. Let the motion of exercise 8, page 162, be retarded by a damping force of magnitude $0.6|v|$. Find the equation of motion. ANS. $x = 0.30\,e^{-4.8t}\cos 6.4t + 0.22\,e^{-4.8t}\sin 6.4t - 0.05\cos 8t$ (ft).

14. Show that whenever $t > 1$ (sec), the solution of exercise 13 can be replaced (to the nearest 0.01 ft) by $x = -0.05\cos 8t$.

15. Let the motion of exercise 8, page 162, be retarded by a damping force of magnitude $|v|$. Find the equation of motion and also determine its form (to the nearest 0.01 ft) for $t > 1$ (sec). ANS. $x = \frac{9}{32}(8t + 1)\,e^{-8t} - \frac{1}{32}\cos 8t$ (ft); for $t > 1$, $x = -\frac{1}{32}\cos 8t$.

16. Let the motion of exercise 8, page 162, be retarded by a damping force of magnitude $\frac{5}{3}|v|$. Find the equation of motion. ANS. $x = 0.30\,e^{-(8/3)t} - 0.03\,e^{-24t} - 0.02\cos 8t$.

17. Alter exercise 6, page 162, by inserting a damping force of magnitude one-half that of the velocity and then determine x. ANS. $x = \exp(-\frac{2}{3}t)(0.30\cos 8.0t - 0.22\sin 8.0t) - 0.05\cos 4t + 0.49\sin 4t$.

18. A spring is stretched 6 in. by a 4 lb weight. Let the weight be pulled down 6 in. below equilibrium and given an initial upward velocity of 7 ft/sec. Assuming a damping force twice the magnitude of the velocity, describe the motion and sketch the graph at intervals of 0.05 sec for $0 \leq t \leq 0.3$ (sec). ANS. $x = \frac{1}{2}e^{-8t}(1 - 6t)$.

19. An object weighing w lb is dropped from a height h ft above the earth. At time t (sec) after the object is dropped, let its distance from the starting point be x (ft), measured positive downward. Assuming air resistance to be negligible, show that x must satisfy the equation

$$\frac{w}{g}\frac{d^2x}{dt^2} = w$$

as long as $x < h$. Find x. ANS. $x = \frac{1}{2}gt^2$.

20. Let the weight of exercise 19 be given an initial velocity v_0. Let v be the velocity at time t. Determine v and x. ANS. $v = gt + v_0$, $x = \frac{1}{2}gt^2 + v_0t$.

21. From the results in exercise 20, find a relation that does not contain t explicitly. ANS. $v^2 = v_0^2 + 2gx$.

22. If air resistance furnishes an additional force proportional to the velocity in the motion studied in exercises 19 and 20, show that the equation of motion becomes

$$\frac{w}{g}\frac{d^2x}{dt^2} + b\frac{dx}{dt} = w. \qquad \text{(A)}$$

Solve equation (A) given the conditions $t = 0, x = 0$, and $v = v_0$. Use $a = bg/w$.

ANS. $x = a^{-1}gt + a^{-2}(av_0 - g)(1 - e^{-at})$.

23. To compare the results of exercises 20 and 22 when $a = bg/w$ is small, use the power series for e^{-at} in the answer for exercise 22 and discard all terms involving a^n for $n \geq 3$. ANS. $x = \frac{1}{2}gt^2 + v_0 t - \frac{1}{6}at^2(3v_0 + gt) + \frac{1}{24}a^2t^3(4v_0 + gt)$.

24. The equation of motion of the vertical fall of a man with a parachute may be roughly approximated by equation (A) of exercise 22. Suppose a 180-lb man drops from a great height and attains a velocity of 20 miles per hour (mph) after a long time. Determine the implied coefficient b of equation (A). ANS. 6.1 (lb)(sec)/ft.

25. A particle is moving along the x-axis according to the law

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0.$$

If the particle started at $x = 0$ with an initial velocity of 12 ft/sec to the left, determine: (a) x in terms of t; (b) the times at which stops occur; and (c) the ratio between the numerical values of x at successive stops.

ANS. (a) $x = -3\,e^{-3t}\sin 4t$.
(b) $t = 0.23 + \frac{1}{4}n\pi, n = 0, 1, 2, 3, \ldots$.
(c) 0.095.

55. The simple pendulum

A rod of length C ft is suspended by one end so it can swing freely in a vertical plane. Let a weight B (the bob) of w lb be attached to the free end of the rod, and let the weight of the rod be negligible compared to the weight of the bob.

Let θ (radians) be the angular displacement from the vertical, as shown in Figure 17, of the rod at time t (sec). The tangential component of the force

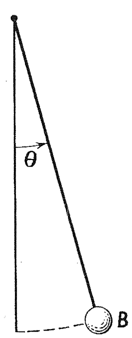

FIGURE 17

w (lb) is $w \sin\theta$ and it tends to decrease θ. Then, neglecting the weight of the rod and using $S = C\theta$ as a measure of arc length from the vertical position, we may conclude that

$$\frac{w}{g}\frac{d^2S}{dt^2} = -w\sin\theta. \tag{1}$$

Since $S = C\theta$ and C is constant, (1) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{C}\sin\theta = 0. \tag{2}$$

The solution of equation (2) is not elementary; it involves an elliptic integral. If θ is small, however, $\sin\theta$ and θ are nearly equal and (2) is closely approximated by the much simpler equation

$$\frac{d^2\theta}{dt^2} + \beta^2\theta = 0; \qquad \beta^2 = \frac{g}{C}. \tag{3}$$

The solution of (3) with pertinent initial conditions gives usable results whenever those conditions are such that θ remains small, say $|\theta| < 0.3$ (radians).

Exercises

1. A clock has a 6-in. pendulum. The clock ticks once for each time that the pendulum completes a swing, returning to its original position. How many times does the clock tick in 30 sec? ANS. 38 times.

2. A 6-in. pendulum is released from rest at an angle one-tenth of a radian from the vertical. Using $g = 32$ (ft/sec^2), describe the motion. ANS. $\theta = 0.1\cos 8t$ (radians).

3. For the pendulum of exercise 2, find the maximum angular speed and its first time of occurrence. ANS. 0.8 (radians/sec) at 0.2 sec.

4. A 6-in. pendulum is started with a velocity of 1 radian/sec, toward the vertical, from a position one-tenth radian from the vertical. Describe the motion. ANS. $\theta = \frac{1}{10}\cos 8t - \frac{1}{8}\sin 8t$ (radians).

5. For exercise 4, find to the nearest degree the maximum angular displacement from the vertical. ANS. 9°.

6. Interpret as a pendulum problem and solve:

$$\frac{d^2\theta}{dt^2} + \beta^2\theta = 0 : \beta^2 = \frac{g}{C}, \qquad \text{when } t = 0, \theta = \theta_0, \omega = \frac{d\theta}{dt} = \omega_0.$$

ANS. $\theta = \theta_0 \cos\beta t + \beta^{-1}\omega_0 \sin\beta t$ (radians).

7. Find the maximum angular displacement from the vertical for the pendulum of exercise 6. ANS. $\theta_{max} = (\theta_0^2 + \beta^{-2}\omega_0^2)^{1/2}$.