Math 5331, Analysis I, Homework Supplement

Base β and Cantor sets

1. The representation of numbers in base β

Let n be an integer, say $n = 31203$. This can be expressed equivalently as

$$31204 = 3 * 10^4 + 1 * 10^3 + 2 * 10^2 + 0 * 10^1 + 4 * 10^0.$$

This is a base 10 expansion, and we write $(31204)_{10}$ if we want to emphasize the point. One can also deal with decimals. For example,

$$(23.5619)_{10} = 2 * 10 + 3 * 10^0 + 5 * 10^{-1} + 6 * 10^{-2} + 1 * 10^{-3} + 9 * 10^{-4}.$$

But there are other bases of interest, such as base $\beta = 2$, the binary numbers. Here,

$$(110.0101)_2 = 1 * 2^2 + 1 * 2^1 + 0 * 2^0 + 0 * 2^{-1} + 1 * 2^{-2} + 0 * 2^{-3} + 1 * 2^{-4}.$$

In general, we write

$$(a_k \dots a_0.a_{-1} \dots a_{-m})_\beta = a_k * \beta^k + \dots + a_0 * \beta^0 + a_{-1} * \beta^{-1} + \dots + a_{-m} * \beta^{-m},$$

where $0 \leq a_i < \beta$, $a_k \neq 0$. In a sense, we are converting base β to what we are most familiar with, base 10.

Exercise 1.1. Exercise 1: Convert $(978.23)_{10}$, $(1101.101)_2$ and $(120.221)_3$ to base 10.

Recall the geometric series:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad \text{if } |r| < 1.$$

Exercise 1.2. Convert $(.2\overline{2})_{10}$, $(.9\overline{9})_{10}$, $(.1\overline{1})_2$, $(.1\overline{1})_3$ and $(.2\overline{2})_3$ to base 10.

2. The Cantor middle thirds set

The Cantor middle-thirds set is constructed by intersecting an infinite nested sequence of closed sets, each described by removing the middle third of it's predecessor. Explicitly, $C = \cap_{i=0}^{\infty} C_i$ with

$$C_0 = [0,1]$$
$$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$$
$$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{3}{9}] \cup [\frac{6}{9},\frac{7}{9}] \cup [\frac{8}{9},\frac{9}{9}]$$
$$\vdots$$

Note that each C_i is closed and nonempty, and $C_{i+1} \subset C_i$. And so by the nested interal theorem C is closed and non-empty.

Exercise 2.1. Show that the 1-dimensional length of the Cantor set is 0. (Hint: Consider it's complement, i.e., what was removed).

In particular, the Cantor set cannot contain any real interval (a, b). Moreover, the Cantor set contains exactly the end points of each interval of each set C_i. Any element of the Cantor set is of the form $(.a_{-1}a_{-2}a_{-3}\ldots)_3$ where $a_i = 0$ or 2. That is,

$$\frac{a_{-1}}{3} + \frac{a_{-2}}{3^2} + \frac{a_{-3}}{3^3} + \cdots$$

Exercise 2.2. Find the base 3 representation for the end points of the intervals in C_0, C_1, C_2 using only the digits 0 and 2 (recall that $(.2\overline{2})_3 = 1$).

Even though one can easily list the end points in C_0, C_1, ..., the Cantor set C is uncountable. This is very unintuitive. Here's the argument, a so-called "diagonal" argument.

Theorem 2.3. *Cantor's middle-thirds set is uncountable.*

Proof. (Proof 1) Assume that the elements in C can be counted. Then, we can list them x_1, x_2, x_3, ... In base 3 these can be represented

$$x_1 = .a_{11}a_{12}a_{13}a_{14}\ldots$$

$$x_2 = .a_{21}a_{22}a_{23}a_{24}\ldots$$

$$x_3 = .a_{31}a_{32}a_{33}a_{34}\ldots$$

Now, consider the number $x = .a_1a_2a_3\ldots$ defined such that $a_i = 0$ if $a_{ii} = 2$, and $a_i = 2$ if $a_{ii} = 0$. For example, if $x_1 = (.022\ldots)_3$, $x_2 = (.200\ldots)_3$ and $x_3 = (.202\ldots)_3$, ..., then we define $x = (.220\ldots)_3$. In particular, x was not included among the x_i, and so the countability assumption is false. Hence, C is uncountable. □

Proof. (Proof 2) As above, every $x \in C$ can be represented by an infinite base 3 expansion with entries either 0 or 2. There is a 1-1 correspondence with such a representation and an element of the Cantor set. Now, change all $2's$ to $1's$. For example, $(.20220\ldots)_3$ becomes $(.10110\ldots)_2$. This second number is a unique base 2 representation for a number in $[0, 1]$. Moreover, any number in $[0, 1]$ can be represented this way. Hence, we have a 1-1 correspondence between elements of the Cantor set and numbers in $[0, 1]$. But since $[0, 1]$ is uncountable, then so is C. □

We have shown that the Cantor middle-thirds set C is closed, uncountable, and has 1-dimensional length 0. It is however possible to construct other Cantor-like sets, some with positive length.

Exercise 2.4. Construct a Cantor-like set (the "fat Cantor set") by removing the middle interval of length 1/4 from $F_0 := [0, 1]$ to get F_1. Then the middle intervals of length 1/16 from F_1 to get F_2. Then the middle intervals of length 1/64 from F_2 to get F_3. And so on. At the $n - th$ step in the constuction F_n consists of 2^n subintervals of F_{n-1}. Prove that $F = \cap F_n$ has 1-dimensional length greater than zero (find this length).