Math 5530 Review Test I

Read each question carefully. Avoid making simple mistakes. Use the back of the page if necessary. You must show your work in order to receive full credits.

(1) A differential equation is an equation involving derivatives or differentials. Determine the type of the following equations by indicating the order, ODE/PDE, linear/nonlinear. If linear, tell if it is homogeneous or inhomogeneous.

a $(y'')^2 - 6x = (y')^3$
b $y' = \frac{a\cos x + b\sin y}{a\sin x + b\cos y}$ $\quad$ (a, b are constants)
c $\frac{d^2y}{dt^2} + 13\frac{dy}{dt} + 36y = 4e^t$
d $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})U = 0$
e $y = xy' - y'^2$
f $(x^2 - x)dy = (2x-1)ydx$
g $y' = x^2 + y^2$
h $u_t = \Delta u + f(t, x)$ $\quad$ (heat equation)
i $u_t + u_{xxx} + 6uu_x = 0$
j $u_t = \frac{\partial}{\partial x}F(u, u_x)$
k $u_t + uu_x = 0$ $\quad$ (Burgers equation)
l $\nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0$ $\quad$ (minimal surface equation)

(2) Find the solution to the initial value problem

$$x' = x\sin t + 2te^{-cost}, \quad x(0) = 1$$

(3) Determine if the equation is exact and solve it if it is.

$$(2x\sin y + 3x^2y)dx + (x^3 + x^2\cos y + y^2)dy = 0$$

(4) The ODE $-ydx + xdy = 0$ is not exact. Multiply by $1/x^2$ will make it exact. some other integrating factors are $1/y^2, 1/(xy), 1/(x^2+y^2)$. In general, given $Mdx + Ndy = 0$, by Theorem 1.4 and 1.5 in Section 1.4 (E. Kreyszig):

(a) If $R(x) := \frac{1}{N}(M_y - N_x)$ depends on x only, then the integrating factor

$$\mu(x) = e^{\int Rdx}$$

(b) If $R^*(x) := \frac{1}{M}(N_x - M_y)$ depends on y only, then the integrating factor

$$\mu(y) = e^{\int R^*dy}$$

Solve $(x^2 + y^2)dx - 2xydy = 0$ [Hint: #5 in [Kreyszig Section 1.4]]

(5) Solve the equations. Determine if the differential equations are homogeneous. If so, determine its degree.

(a)

$$ydx + (y - x)dy = 0$$

(b) $xy' = y + 3x^4\cos^2(y/x)$, $y(1) = 0$. [Clue: substitution $y = xu$, [Kreyszig, Section 1.3, # 17]]

(6) * Find a general solution of
$$\frac{dy}{dx} = 6\frac{y}{x} - xy^2$$

(7) Find the solutions of $y^{(4)} + 8y'' + 16y = 0$ (answer: $y = c_1 \sin 2t + c_2 \cos 2t + c_3 t \sin 2t + c_4 t \cos 2t$)

(8) * Find the orthogonal trajectories of the family of curves
  (a) $xy = c$
  (b) $x^2 + y^2 = cx$
  (c) $y^2 = cx^2 - 2y$

(9) Let $D = d/dx$. Solve the Cauchy-Euler equation $(x^2D^2 + xD - 4)y = x^3$ [Clue I: change of variable $x = e^t$; clue II: let $y = x^k$]

(10) The operator $L := a_0(x)D^2 + a_1(x)D + a_2(x)$ is exact $\iff$ $a_0'' - a_1' + a_2 = 0$, in which case
$$Ly = (a_0D^2 + a_1D + a_2)y = D(a_0D + a_1 - a_0')y$$
Find the solution of $(1 - x^2)y'' - 3xy' - y = 1$.

(11) Figure 1 (Page 4) is the direction field for the differential equation $y' = y(y - 1)(y + 1)$.
  (a) Draw on the direction field the solutions of the differential equation satisfying each of the following initial values.
    (i) $y(0) = 0.0$
    (ii) $y(0) = 0.5$
    (iii) $y(0) = -1.5$
  (b) For the solution $y(t)$ with initial condition $y(0) = 0.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?
  (c) For the solution $y(t)$ with initial condition $y(0) = -1.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

(12) Figure 2 (Page 4) is the direction field for the differential equation $y' = y - t$.
  (a) Draw on the direction field the solutions of the differential equation satisfying each of the following initial values.
    (i) $y(0) = 0.0$
    (ii) $y(0) = 1.0$
    (iii) $y(0) = -1.0$
    (iv) $y(0) = 2.0$
  (b) Are there any constant solutions $y = c$ to this differential equation? If so, show them on the direction field.
  (c) Are there any straight line solutions $y = mt+b$? If so indicate them on the direction field.
  (d) There is a number c such that all solutions with initial condition $y(0) > c$ satisfy $\lim_{t\to\infty} = \infty$ and all solutions with initial condition $y(0) < c$ satisfy $\lim_{t\to\infty} = -\infty$. Find this number c by inspecting the direction field.

Figure 1. Direction Field for Exercise 11

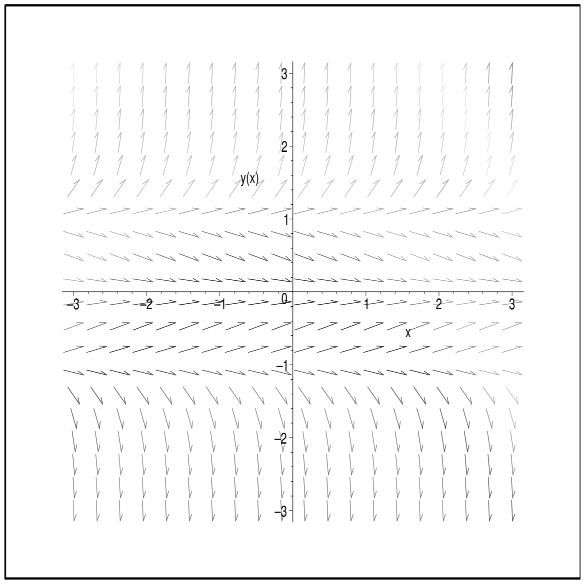
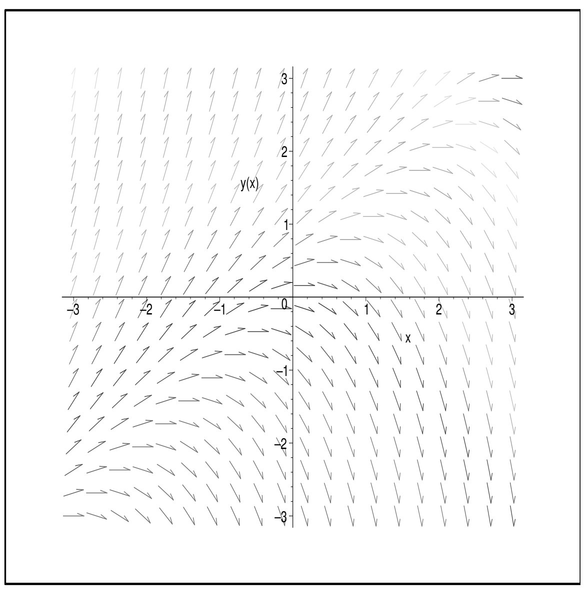


Figure 2. Direction Field for Exercise 12



(13) Solve each of the following initial value problems. You **must** show your work to tell if they are unique?
(a) $y' = \pm y^{-3}, \quad y(1) = -1.$
(b) $y' = |y|^{2/3}, \quad y(t_0) = y_0$
(c) $y' + \dfrac{3}{t}y = 7t^3, \quad y(1) = -1.$

(14) [# 5, Kreyszig, Section 2.4] What are the frequencies of vibration of a body of mass $m = 5$ kg

(a) on a spring of modulus $k_1 = 20nt/m$
(b) on a spring of modulus $k_2 = 45nt/m$
(c) on the two springs in parallel?

(15) [# 7, Kreyszig, Section 2.4] Find the frequency of oscillation of a pendulum of mass m and of length L, neglecting air resistance and the weight of the rod, and assuming the angle θ to be so small that $\sin\theta$ practically equals θ.

(16)] [Ex.2, Sec. 2.4, Kreyszig] Consider the damped system $my" + cy' + ky = 0$ with IC $y(0) = 0.16m, y'(0) = 0$ where $m = 10, k = 90$ under the following conditions
(a) $c = 100$kg/sec,
(b) $c = 60$kg/sec,
(c) $c = 10$kg/sec.
[Clue: a) $y = -0.02e^{-9t} + 0.18e^{-t}$ (overdamping)
b) $y = (0.16 + 0.48t)e^{-3t}$ (critical damping)
c) $y = e^{-t/2}(0.16\cos 2.96t + 0.027\sin 2.96t)$ (underdamping)]

Solutions

2. This is first order linear ODE $x' + Px = Q$, where $P = -\sin t, Q = 2te^{-cost}$. The general formula gives

$$\begin{aligned} x(t) &= e^{-\int P}\int e^{\int P}Qdt = e^{\int(\sin t)}\int e^{\int(-\sin t)}2te^{-cost}dt \\ &= e^{\int(\sin t)}\int e^{\cos t}2te^{-cost}dt = e^{\int(\sin t)}\int 2tdt \\ &= e^{-cost}(t^2 + C) \end{aligned}$$

Now plugging in $t = 0, x = 1$ to obtain $C = e$.

3. It is Exact by the following test: The differential form $Mdx + Ndy = 0$ is exact $\iff$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Since it is exact, there exists $f(x, y)$ such that $df = f_x dx + f_y dy = Mdx + Ndy$. We will solve f to obtain the equation $f(x, y) = C$ which implicitly defines the solution of

$$(2x\sin y + 3x^2 y)dx + (x^3 + x^2\cos y + y^2)dy = 0$$

From $\frac{\partial f}{\partial x} = 2x\sin y + 3x^2 y$ we get

$$f(x,y) = \int (2x\sin y + 3x^2y)dx = x^2\sin y + x^3y + C(y)$$

Taking derivative in y of the above yields

$$\begin{aligned}&\partial_y f(x,y) = \partial_y(x^2\sin y + x^3 y + C(y))\\ =&x^2\cos y + x^3 + C'(y) = N = x^3 + x^2\cos y + y^2\end{aligned}$$

which suggests $C'(y) = y^2 \to C(y) = y^3/3$. Hence we arrive at the equation

$$f(x,y) = x^2\sin y + x^3 y + y^3/3 = C.$$

6*. This is first-order quadratic equation (Bernoulli type). $n = 2$ Substitution $w = y^{1-n} = y^{-1} \to y = w^{-1}$. We have $\frac{dy}{dx} = -w^{-2}\frac{dw}{dx}$ and so

$$\begin{aligned}&-w^{-2}\frac{dw}{dx} = 6\frac{w^{-1}}{x} - xw^{-2}\\ &(\text{multiplying } -w^2 \text{ both sides} \to)\quad \frac{dw}{dx} = -6\frac{w}{x} + x\end{aligned}$$

This is a 1st-order ODE, you can solve to get $w = w(x)$ and then replace w by y^{-1} and then simplify to obtain the solution $y = y(x)$.

Indeed,

$$\begin{aligned}&w = w(x) = e^{-\int \frac{6}{x}}\left(\int e^{\int \frac{6}{x}} x dx\right)\\ =&e^{-6\ln|x|}\left(\int x^6 x dx\right) = x^{-6}(x^8/8 + C)\\ =&x^2/8 + Cx^{-6}.\end{aligned}$$

From this we obtain $y = \frac{1}{x^2/8 + Cx^{-6}}$.

7 Solve the characteristic equation

$$\begin{aligned}&r^4 + 8r^2 + 16 = 0\\ &(r^2+4)^2 = 0\\ &r_{1,2} = \pm 2i, \quad r_{3,4} = \pm 2i.\end{aligned}$$

8 Clue: (a) Consider $F(x,y) = xy$, then $xy = C$ are family of level curves for F. The gradient $\nabla F = \langle y, x\rangle$ will be normal to these level curves. Hence, following the slope field method the orthogonal trajectories satisfy

$$\frac{dy}{dx} = f(x,y) = \frac{F_y}{F_x}$$

from which we can solve for $y = y(x)$.

10 $a_0 = 1 - x^2$, $a_1 = -3x, a_2 = -1$, we find $a_0'' - a_1' + a_2 = -2 - (-3) + (-1) = 0 \Longrightarrow$ $D((1-x^2)D - 3x + 2x)y = D\left(((1-x^2)D - x)y\right) = 1$

$$\begin{aligned}((1-x^2)D - x)y &= \int 1dx = x + C_1\\ (1-x^2)Dy - xy &= x + C_1\\ Dy - \frac{x}{1-x^2}y &= \frac{x+C_1}{1-x^2}\end{aligned}$$

$\square$

11 a) A direction field or slope field shows the directions of the family of solution curves, which can be obtained by drawing short line-segments in the xy-plane based on the ODE

$$y' = f(x, y)$$

where the slope of the line segment at (x, y) is given exactly by $f(x, y)$. Plot the solution curves that fit this field by following the flows or "'trajectories" in the direction field passing (i) $(0, 0)$; (ii) $(0, 0.5)$; (iii) $(0, -1.5)$.

b) $\lim_{t\to\infty} y(t) = 0$; $\lim_{t\to-\infty} y(t) = 1$

c) $\lim_{t\to\infty} y(t) = -\infty$; $\lim_{t\to-\infty} y(t) = -1$

12 a) A direction field is made up of short line-segments indicating the slopes of the tangent lines of the solution curve at each coordinate points within its domain. Typically we can organize the line-segments using either grid method or level sets method. One can use the direction field to sketch (approximate) solution curves through specific points in the field by following those small lines in the direction field.

b) No constant solution.

c) Yes. Straight line solution is given by $y = t + 1$.

d) $c = 1$.

13 a) For fixed idea, let us consider $y' = y^{-3}$, $y(1) = -1$. Apply separation of variable method to obtain $y(t) = \pm(C + 4t)^{1/4}$. Also the I.C. requires $C = -3$. Hence $y = -(4t-3)^{1/4}$. We can tell that there exists one and only one solution on $(\frac{3}{4}, \infty)$.

b) Divided into two cases $y > 0$ and $y < 0$. In either case we find that the general solution is given by $y = (\frac{t}{3} + C)^3$.

Substitute the point (t_0, y_0) to determine $C = y_0^{1/3} - \frac{t_0}{3}$. Hence the solution shows that there always exists one and only one solution, even though the function $f(x, y) = |y|^{2/3} = y^{2/3}$ does not have a continuous partial derivative at $(t_0, 0)$!

c) This is 1st-order linear equation. The E and U theorem for 1st-order **linear** ode tells that, since the coefficients $p = 3/t$ and $Q = 7t^3$ are continuous on an interval $I = (0, \infty)$ and $(-\infty, 0)$, there exists a unique solution on I.

We can verify this by solving the ode to find that $y = t^4 + \frac{C}{t^3}$. Sub. the I.C. to obtain $C = -2$. Hence $y = t^4 - \frac{2}{t^3}$, which verifies the E and U assertion.

16 Solve the characteristic equation $mr^2 + cr + k = 0$. from which we can obtain fundamental set

(a) $\{e^{r_1 t}, e^{r_2 t}\}$ if $r_1 \neq r_2$;

(b) $\{e^{r_1 t}, te^{r_1 t}\}$ if $r_1 = r_2$;
(c) $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ if r_1, r_2 complex conjugate $\alpha + i\beta$.