Project 3 (review test)
Math 5530

Name
Id
Section

Read each problem carefully. Avoid simple mistakes. To receive full credits you must show your work to support and justify your answer. An incomplete answer might receive partial credits if you have written down a reasonable solution. Questions marked with (*) are optional.

1. (a) Find the general solution to 1^{st}-order linear ODE

$$x' - x\sin t = 2te^{-\cos t}$$

   (b) Then find the solution satisfying the initial condition $x(0) = 1$, and, $x(\pi) = -1$ respectively.

2. Figure 1 (Page 2) is the direction field for $y' = y(y-1)(y+1)$.

   (a) Draw on the direction field the solutions of the differential equation satisfying each of the following initial values.

      i. $y(0) = 0.0$
      ii. $y(0) = 0.5$
      iii. $y(0) = -1.5$

   (b) For the solution $y(t)$ with initial condition $y(0) = 0.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

   (c) For the solution $y(t)$ with initial condition $y(0) = -1.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

   (d) The constant solutions $y = 0$, $y = 1$, $y = -1$ are three equilibrium solutions. Which one is stable? asymptotically stable and/or unstable?

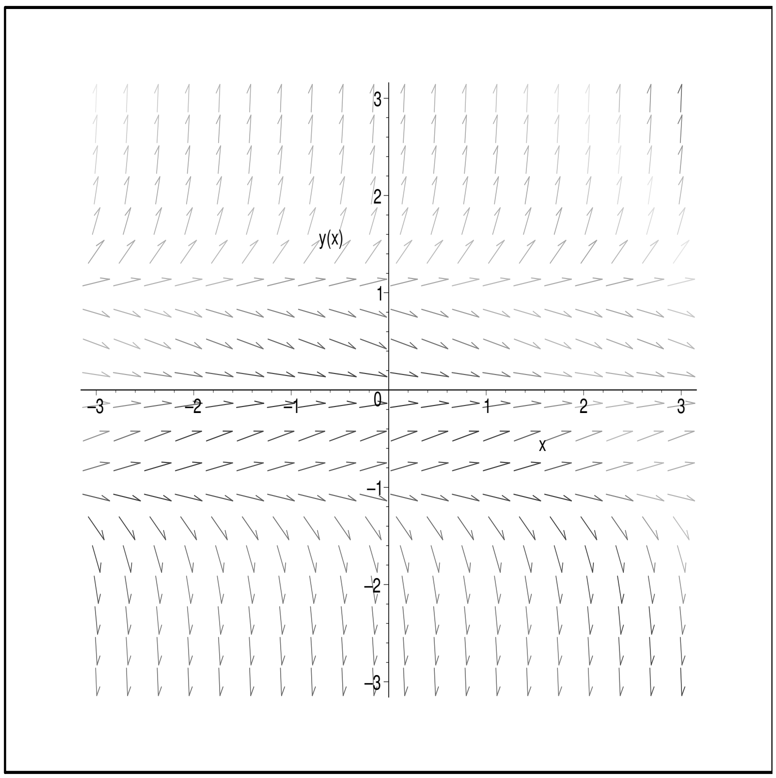
3. a) Given an example of a set of three functions so that they are Linear Independent.
b) Given an example of a set of three functions so that they are Linear Dependent.
c) Suppose $p(t), q(t)$ and $f(t)$ are ________on an open interval (a, b) containing $t = t_0$. Then the IVP

$$y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = y_0, y'(t_0) = y_1$$

has a ________ solution on (a, b)
d*) Use the Existence and Uniqueness Theorem for linear IVP (Initial Value Problem) to determine the largest interval on which the solution is guaranteed to exist.

$$y'' + \frac{y'}{x-5} + \frac{y}{(x-5)^2} = \frac{\sin x}{2x+3}, \; y(0) = \pi, \; y'(0) = 0.5\pi$$

Figure 1: Direction Field for Problem 2



[Hint to Ex.3: A set of functions $\{u_1, u_2, \ldots, u_k\}$ are called linearly independent provided the only solution to

$$c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = 0 \tag{1}$$

is $c_1 = c_2 = \cdots = c_k = 0$. The set $\{u_1, u_2, \ldots, u_k\}$ are called linearly dependent if there exists some solution to (1) such that not all c_i are zero, $1 \leq i \leq k$.

Alternatively, you could also compute the Wronskian

$$W[u_1, \cdots, u_k] = det \begin{pmatrix} u_1 & u_2 & \cdots & u_k \\ u_1' & u_2' & \cdots & u_k' \\ & \cdots & & \\ u_1^{(k-1)} & u_2^{(k-1)} & \cdots & u_k^{(k-1)} \end{pmatrix}.$$

If $W[u_1, \cdots, u_k] \neq 0$, then $u_1, \ldots, u_k$ is linear independent; otherwise, then dependent.

For the E and U questions on (c) and (d), please refer to [Kreyszig, Section 2.6, p.74]; basically you want $p(t)$, $q(t)$ and $f(t)$ to be continuous on a common (maximal) interval.]

4. Calculate the laplacian $\nabla^2 f = \Delta f = \text{div}\,(\nabla f)$, where $f(x, y) = \ln(x^2 + y^2)$. What is the domain of Δf? (Clue: [Kreyszig, Section 9.8, #17])

5. Find the Fourier series of the function $f(x) = |x|$, which is assumed to have the period 2π. Show detailed steps in your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$. (Clue: [Kreyszig, Section 12.1, #12])

6. Solve the Dirichlet problem for the wave equation in $[0, \pi]$.
$u_{tt} - 9u_{xx} = 0$ in $(t, x) \in (-\infty, \infty) \times (0, \pi)$
$u(0, x) = \cos x,\ u_t(0, x) = 0$ (initial condition)
$u(t, 0) = u(t, \pi) = 0.$ (boundary condition)

   (Clue: Separation of variables, Fourier series with odd extension to $[-\pi, \pi]$)

7* Find the solution $u(t, x)$ to the diffusion problem with Neumann boundary condition, where the given functions g, h and ϕ are continuous.

$$\begin{aligned}
&u_t - ku_{xx} = 0, \qquad (t, x) \in (0, \infty) \times (0, \ell)\\
&u_x(t, 0) = g(t),\ u_x(t, \ell) = h(t) \quad (\textit{boundary condition})\\
&u(0, x) = \phi(x) \qquad (\textit{initial condition})
\end{aligned}$$

(Clue: Step 1. Separate the variables by writing $u_n = T_n(t)X_n(x)$.
Step 2. Use boundary condition to determine the eigenvalues λ_n.
Step 3. Superposition $u = \sum_n u_n(t, x)$.
Step 4. Apply Fourier series to find the coefficients.)

8 Determine whether $\Phi(t) = \begin{pmatrix} \cos t + 2\sin t & 2\cos t - \sin t \\ \sin t & \cos t \end{pmatrix}$ is a fundamental matrix for the linear system $X'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X(t)$. If so, give an expression of a general solution.

[Hint: If $X(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_3(t) & x_4(t) \end{pmatrix}$, then $X'(t) = \begin{pmatrix} x_1'(t) & x_2'(t) \\ x_3'(t) & x_4'(t) \end{pmatrix}$. Using this definition, plug in $X = \Phi(t)$ on the two sides of the equation $X' = AX$. If one can verify that equation, then $X_1(t) = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$ and $X_2(t) = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$ will be two solutions to the system $X' = AX$. If $det(\Phi(t)) \neq 0$, or X_1 and X_2 are two linearly independent solutions, then $\Phi(t)$ is a fundamental matrix. see [Kreyszig, Section 4.2, p.139]. The general solution is then given by $X = c_1X_1 + c_2X_2$.]