Math 5539 - Mathematical Models - Midterm 1 Study Guide

Name ______________________

Eagle ID ______________________

Instructions: This test has a total of 40 points. Read each question carefully. Avoid making simple mistakes. You must show your work to support and justify the answer in order to receive full credits.

(1) Consider

$$m\frac{d^2y}{dt^2} = \underbrace{-c\frac{dy}{dt}}_{friction} \underbrace{-mg}_{gravitational\ force}.$$

Assume $m = 2$ and $c = 3$.
This differential equation models a free-falling body with the frictional force proportional to the velocity. Find the free-falling body's terminal velocity. You must show all your work to receive full credit
(Hint: First, let $v(t) = \frac{dy}{dt}$ and solve the differential equation to find $v(t)$. Then let $t \to \infty$ to find the terminal velocity.)

Solution) Consider

$$m\frac{d^2y}{dt^2} = \underbrace{-c\frac{dy}{dt}}_{friction} \underbrace{-mg}_{gravitational\ force}, \quad c > 0.$$

The above differential equation can be solvable using the method of undetermined coefficients. However, one can quickly solve the differential equation using the following approach:
Let $v = \frac{dy}{dt}$. Then

$$\begin{aligned}
& m\frac{d^2y}{dt^2} = -c\frac{dy}{dt} - mg \\
& \Rightarrow m\frac{dv}{dt} = -cv - mg \\
& \Rightarrow \frac{m}{cv + mg}dv = -dt \\
& \Rightarrow \int_{v_0}^{v} \frac{m}{cv + mg}dv = \int_0^t -dt \\
& \Rightarrow \left[\frac{m}{c}\ln(cv + mg)\right]_{v_0}^{v} = -t \\
& \Rightarrow \frac{m}{c}\ln(cv + mg) - \frac{m}{c}\ln(cv_0 + mg) = -t \\
& \Rightarrow \ln(cv + mg) = -\frac{c}{m}t + \ln(cv_0 + mg) \\
& \Rightarrow cv + mg = (cv_0 + mg)e^{-\frac{c}{m}t}
\end{aligned}$$

Therefore, we obtain

$$v(t) = -\frac{mg}{c} + (v_0 + \frac{mg}{c})e^{-\frac{c}{m}t}.$$

As $t \to \infty$, we have

$$v(t) \to \underbrace{-\frac{mg}{c}}_{terminal\ velocity}.$$

Since $m = 2$ and $c = 3$, we obtain

$$v(t) \to \underbrace{-\frac{2g}{3}}_{terminal\ velocity}.$$

(2) Consider a nonlinear pendulum equation:

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta.$$

Assume that a nonlinear pendulum is initially at $\theta_0 = \frac{\pi}{2} = 90\ degree$.

(a) Formulate conservation of energy.(You don't need to show your work)
Solution)
Energy conservation:

$$\frac{L}{2}\left(\frac{d\theta}{dt}\right)^2 + g(1-\cos\theta) = E.$$

Here, the total energy E is determined by initial conditions $\left.\frac{d\theta}{dt}\right|_{t=0} = v_0$ and $\theta(0) = \theta_0$:

$$E = \frac{L}{2}v_0^2 + g(1-\cos\theta_0).$$

By the assumption in the problem, we have $\theta_0 = \frac{\pi}{2}$. So

$$E = \frac{L}{2}v_0^2 + g. \tag{1}$$

(b) How large an initial angular velocity is necessary for the pendulum to go completely around?
Solution)
From Sections 23 and 24, we know that if

$$E = 2g,$$

then the pendulum converges to π but never reaches π. On the other hand, if

$$E > 2g, \tag{2}$$

then there is more energy than is need for the pendulum to almost go around so the pendulum will complete one cycle around and continue revolving.

By collecting (1) and (2), we obtain

$$v_0^2 > \frac{2g}{L}.$$

This implies that if an initial angular velocity is

$$v_0 > \sqrt{\frac{2g}{L}}, \quad or \quad v_0 < -\sqrt{\frac{2g}{L}},$$

then $E > 2g$ and the pendulum will go completely around.

(3) A linear spring-mass system (without friction) satisfies

$$4 \cdot \frac{d^2x}{dt^2} = -3x.$$

Consider the initial value problem such that at $t = 0$,

$$x_0 = 2, \quad and \quad \frac{dx}{dt} = v_0 = 3.$$

(a) Formulate conservation of energy and evaluate E(total energy). (You don't need to show your work)
Solution)

$$2 \cdot \left(\frac{dx}{dt}\right)^2 + \frac{3}{2}x^2 = E,$$

where

$$E = 2 \cdot (3)^2 + \frac{3}{2} \cdot (2)^2 = 24.$$

(b) What are the velocities of the mass when it passes its equilibrium position?
Solution)
Note that the equilibrium position is $x_E = 0$. At $x_E = 0$, from the energy equation we have

$$2 \cdot \left(\frac{dx}{dt}\right)^2 + \frac{3}{2}(x_E)^2 = E,$$

or

$$2 \cdot \left(\frac{dx}{dt}\right)^2 + 0 = 24.$$

So we obtain the velocities at x_E:

$$v = +\sqrt{12}, \quad or \quad v = -\sqrt{12}.$$

(4) Suppose

$$2 \cdot \frac{d^2x}{dt^2} = -8e^{4x}.$$

(a) Formulate conservation of energy. (You don't need to show your work)
Solution)

$$\frac{2}{2}\left(\frac{dx}{dt}\right)^2 + 2e^{4x} = E,$$

where

$$E = v_0^2 + 2e^{4x_0}.$$

(b) Sketch the solution in the phase plane when $E = 4$(total energy). **Specify** x and y intercept(s) and the curve's direction.
Solution)
When $E = 4$, the conservation of energy becomes

$$\left(\frac{dx}{dt}\right)^2 + 2e^{4x} = 4,$$

or

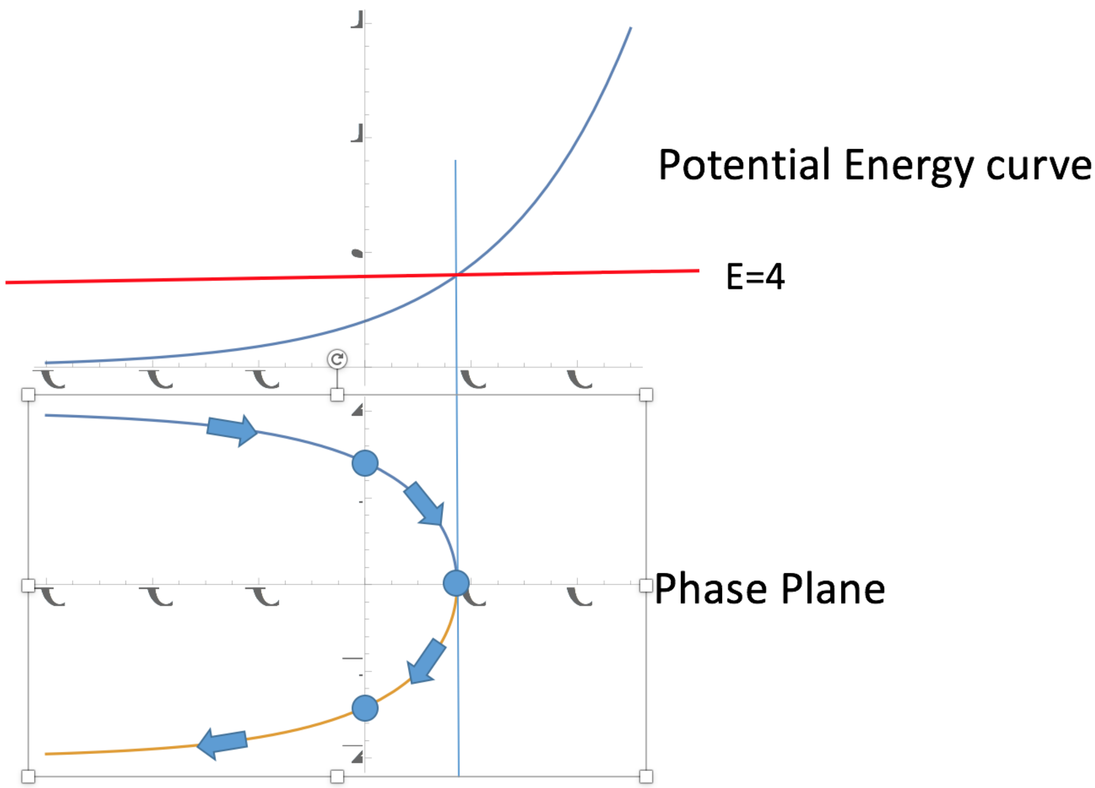
$$\frac{dx}{dt} = \pm\sqrt{4 - 2e^{4x}}.$$

Therefore, the y intercepts are

$$\sqrt{2}, \quad and \quad -\sqrt{2}.$$

x intercept is

$$\frac{1}{4}\ln 2.$$



(c) Suppose that a mass starts at $x = -1$. What is the minimum velocity that the mass can reach $x = 0$?

Solution) In order to have

$$x(t) = 0$$

for some $t > 0$, we need

$$E \geq 2$$

(one can obtain this condition from a phase plane.) From part (a), we have

$$E = v_0^2 + 2e^{4x_0} \geq 2,$$

or

$$v_0^2 + 2e^{4\cdot(-1)} \geq 2.$$

So we obtain

$$v_0 \geq \sqrt{2 - \frac{2}{e^4}}.$$

Therefore, the needed minimum velocity is

$$v_0 = \sqrt{2 - \frac{2}{e^4}}.$$